**Study on “Coincidence Points” Results of**

**“W-D- Mappings of Type - (A)” in CM- Spaces**

 **Abstract**

 In this paper, we obtain a “coincidence point theorem” of W- D(Weakly Decreasing) mappings of ” Type-( A) “ in CM -spaces (Cone- Metric- -Spaces), where the “cone” is not necessarily “normal” cone . Our results are extended, improved and generalized some of the well known results existing in the references.

**Key words**: C-M-Space ( Cone Metric - Space), coincidence point , fixed point and

 W-D(Weakly Decreasing) mappings ofType- (A).

**AMS –Mathematics- Subject- Classification (2010): “54H25” and “47H10”.**

**1. Introduction**

 Huang and Zhang [4] introduced the concept of a cone metric space, where the set of real numbers are replaced by an ordered Banach space and obtained fixed point theorems of contractive type mappings in a CM-space(Cone Metric space). Later on, many authors generalized these fixed point theorems to different types of contractive conditions in cone metric spaces (see, e.g.[2,5,11-13,15]). Ran and Reuring [10] proved existence of fixed point theorems in a partially ordered sets. Since then many authors have generalized their fixed point theorems in different ways (See eg. [9]). Altun et al. [3] introduced the concept of weakly increasing maps and proved some fixed point theorems. G.Jungck [6] introduced the concept of the notion of compatible maps in metric space, while G. jungck[7] introduced the notion of weakly compatible maps in metric spaces. In 2010 S. Janković et al. [8] extended the notion of compatible and weakly compatible maps in cone metric space. In 2012, W. Shatanawi [14] introduce the concept of weakly decreasing maps type (A) and obtained some coincidence point results in cone metric spaces without assuming the normal cone. The result in this paper is an extension of the results 0f [14].

**2. Preliminaries**

 The following are useful in our main results which are due to [4].

**2.1 .Definition.** Let M be a real Banach space and P a subset of M .The set S is called a “cone “

iff

(a). S is non–empty , closed and S ≠ {θ},

(b). α, β ∈ℝ, α, β, x, yϵS implies αx+βy∈S,

(c). -x∈S and x∈S implies “x = θ”.

**2.2.Definition.** Let S be a cone in a “Banach space” M, define partial ordering “ with respect to S by “xy “iff “ y-xϵS”.We shall write “xy” to indicate “xy” but” x ” while “x<<y “will stand for “y-x∈ interior S” , where Interior “S” denotes the interior of the set “S”. This cone “S” is called an “order cone”. It can be easily shown that “λ int(S)int (S)” for all λ∈ℝ+ .

**2.3.Definition.** Let M be a “Banach space” and “S⊂M” be an “order cone” .The order cone “S” is called “normal” if there exists U >0 such that for all x,yM,

 “ θ≼x≼yimplies “║x║≤U║y║”.The least positive number “U” satisfying the this inequality is called the normal constant of “S”.

**2.4.Definition .** Let “X” be a nonempty set of “M” .Suppose that the map

ρ: X X→M satisfies:

 (1). ρ(x, y) for all x, yand ρ(x, y) = θ if and only if x = y ;

 (2).ρ(x, y) = ρ(y, x)  for all x, y;

 (3).ρ(x, y) ρ(x, z) + ρ(y, z) for all x, y, z.

 Then “ρ” is called a “cone metric” on X and (X, ρ) is called a C-M-Space(Cone Metric–Space).

**2.5. Definition.** Let (X, ρ) be a cone metric space .We say that {xn} is said to be

(1). a Cauchy sequence if for every c in M with c >>,there is N such that for all n,m>N,

 ρ(xn, xm) <<c ;

 (2). a convergent sequence if for any “c >> θ”,there is an “N” such that for all n>N,

“ρ(xn, x) << c “, for some fixed “x in X” .We denote this “xnx” (as n.

**2.6. Definition**  .The space (X, ρ) is called a complete C-M-Space if every Cauchy sequence is convergent in it.

**2.7. Definition.** Let (X,) be “partially ordered” set and let “f, T: X→X” be two mappings. We say that “f” is W-D(Weakly -Decreasing) Type –(A) with respect to “T “if the following conditions are hold

(1) fxfy , ), for all xX, and for all y∈T-1(fx).

(2). TX fX.

**2.8. Definition. [8]** Let (X, ρ) be a “C-M-Space” and “A, B: X→X “ be two self-maps. The pair “{A, B}” is said to be” compatible” if, for an arbitrary sequence “{xn}X” such that

 == t X, and for arbitrary cint (P), there exists n0 such that ρ(ABxn, BAxn)<< c whenever n > n0. It is said to be weakly compatible if Ax = Bx implies

 ABx = BAx.

**2.9.Definition [1].** Let  **A,B**: X →X be mapping. If w = Aa = Ba for some a in X, then a is called a “coincidence point” of A and B and “w” is called a “point of coincidence” of”A” and “B”.

**3. The Main Results**

 In this section, we prove our main Theorem.

**3.1. Theorem .**Let (X,) be “partially ordered set” and (X, d) be a “C-M-Space” over a solid cone “P”. Let f, S: X→X be two mappings such that

 d(Sx, Sy)≼α1d(fx, fy)+α2d(fx, Sx)+α3d(fy, Sy)+α4d(fx, Sy)+α5d(fy, Sx) … (1)

for all x, yX, for which “fx” and “fy “ are comparable. And assume that “S” and “f” satisfy the following conditions:

(a). If {xn} is a non - increasing sequence in “X” with respect to such that “xn→x “as

 n→+∞, then “xnx” for all nN.

(b). “ f” is W- D Type –(A) with respect to “S”.

(c). “fX” is complete subspace of “X”.

If α1, α2, α3, α4 , and α5 ≥ 0 with α1+ α2 + α3 + α4 + α5∈[0,1)then “S” and “f” have a coincidence point in X, that is there exists a point “uX “ such that “Su = fu”.

**Proof:** Let “x0 X”. Since “SX fX”, we can choose “x1 X” such that “Sx0 = fx1 “. Also since “SXfX”. Also we choose a point “ x2 X” such that “Sx1 = f x2 ” .Continuing this process, we can construct a sequence “{xn} in X” such that” Sxn= fxn+1 “ .

Since “ xnS-1 (f xn+1)”, nN, then by using the assumption that “f” is “W-D of Type –(A) “ with respect to “S”, we have “fx0 fx1 fx2 … “

By the condition (1) we have,

d(Sxn, Sxn+1 ) ≼ α1d(fxn, fxn+1)+ α2d(fxn, Sxn)+ α3d(fxn+1, Sxn+1)+ α4d(fxn, Sxn+1)

 + α5d(fxn+1, Sxn),

 ≼ α1d(Sxn-1, Sxn)+ α2d(Sxn-1, Sxn)+ α3d(Sxn, Sxn+1)+ α4d(Sxn-1, Sxn+1)

 + α5d(Sxn, Sxn),

 ≼ α1d(Sxn-1, Sxn)+ α2d(Sxn-1, Sxn)+ α3d(Sxn, Sxn+1)+ α4 [d(Sxn-1,Sxn)

 + d(Sxn, Sxn+1)],

 ≼ (α1 + α2+ α4 ) d(Sxn-1, Sxn)+( α3+ α4 )d(Sxn, Sxn+1).

And

 d(Sxn+1, Sxn )≼α1d(fxn+1, fxn)+ α2d(fxn+1, Sxn+1)+ α3d(fxn, Sxn)+ α4d(fxn+1, Sxn)

 + α5d(fxn, Sxn+1),

 ≼α1d(Sxn, Sxn-1)+ α2d(Sxn, Sxn+1)+ α3d(Sxn-1, Sxn)+ α4d(Sxn, Sxn)

 + α5d(Sxn-1, Sxn+1),

 ≼ α1d(Sxn-1, Sxn)+ α2d(Sxn-1,STxn)+ α3d(Sxn, Sxn+1)+ α5 [d(Sxn-1, Sxn)

 + d(Sxn, Sxn+1)],

 ≼ (α1 + α2+ α5 ) d(Sxn-1, Sxn)+(α3+α5 )d(Sxn, Sxn+1).

Hence,

2d(Sxn+1, Sxn) = d(Sxn+1, Sxn)+ d(Sxn, Sxn+1)

 ≼(α1 +α3 +α5 ) d(Sxn, Sxn-1)+ (α2 +α5)d(Sxn, Sxn+1)+(α1+ α2+ α4 ) d(Sxn-1, Sxn)

 +(α3+ α4 )d(Sxn, Sxn+1),

 ≼(2α1+α2+α3 +α4 +α5) d(Sxn-1, Sxn)+( α2 +α3 +α4+ α5 )d(Sxn, Sxn+1),

 (2- α2 - α3 -α4- α5 )d(Sxn+1, Sxn) 2α1+ α2+ α3 +α4 +α5) d(Sxn-1, Sxn)

 d(Sxn-1, Sxn).

Putting, h = . We obtain,

d(Sxn, Sxn+1) h d(Sxn-1, Sxn). … (2)

Thus, for “nN”, we have

d(Sxn, Sxn+1) h d(Sxn-1, Sxn)2 d(Sxn-2, Sxn-1)n d(Sx0, Sx1) .

Let “n, mN” with “m > n”. Then

d(Sxn, Sxm),

Since, h[0,1), we have

d(Sxn, Sxm) ≼ d(Sx0, Sx1) → θ as n→ ∞. … (3)

We shall show that “{Sxn}” is a “Cauchy” sequence in (X, d). For this, let “c >> θ “be given.

Since,” cint(P)”, then there exists a neighborhood of θ, “Nδ(θ) = {yE:║y║< δ}” ,δ > 0, such that

 “c + Nδ(θ) int(P)”. Choose a natural number N1 such that ║d(Sx0, Sx1)║< δ.

Then for all n≥ N1 we have that d(Sx0, Sx1) Nδ(θ).

Hence, c-d(Sx0, Sx1) c + Nδ(θ) int(P).

Thus, we have that for all n≥ N1 ,d(Sx0, Sx1) << c. … (4)

By (3) and (4), it follows that d(Sxn, Sxm) << c whenever n≥ N1 .

Hence, “{Sxn}” is a “Cauchy” sequence in X.

Since, SXfX.

Therefore, “{fxn}” is a “Cauchy” sequence in” fX”. Since, “fX” is complete,

 then there exists “ u = fv “for some “vX” such that” = u = fv”.

Since “{fxn}” is a non–increasing sequence in “X”, then

“fxnfv “for all” nN”, then by (1) we have

d(Sxn, Sv) ≼ α1d(fv, fxn)+α2 d(fxn, Sxn)+ α3d(fv, Sv)+α4d(fxn, Sv)+α5d(fv, Sxn). … (5)

By the triangle inequality and (5) we have

d(fv, Sv)d(fv, fxn) + d(fxn, Sxn)+ d(Sxn, Sv)

 d(fv, fxn)+ d(fxn, Sxn)+ α1d(fv, fxn)+α2 d(fxn, Sxn)+ α3d(fv, Sv)+α4d(fxn, Sv)

 +α5d(fv, Sxn),

 d(fv, fxn)+ d(fxn, Sxn)+ α1d(fv, fxn)+α2 d(fxn, Sxn)+ α3d(fv, Sv)+α4d(fxn, Sv)

 α5 [d(fv,fxn)+(fxn ,Sxn)],

 (1+α1+α5 ) d(fv, fxn)+(1+α2 +α5) d(fxn, Sxn)+ α3d(fv, Sv)+α4d(fxn, Sv),

 (1+α1+α5) d(fv, fxn)+(1+α2 +α5)[d(fxn,fv)+d(fv,Sxn)]+ α3d(fv, Sv)

 +α4[d(fxn, fv)+d(fv,Sv)] ,

 (2+α1+α2+α4+2α5 )d(fv, fxn)+(1+α2 +α5)d(fv,Txn)+( α3+α4 )d(fv, Tv).

Hence, we have

1-( α3+ α4 ) d(fv, Sv) (2+α1+α2 + α4 +2α5)d(fv, fxn)+ (1+α2 +α5) d(fv, Sxn).

d(fv, Sv)(fv, fxn)+ d(fv, Sxn). … (6)

Let “ c>>θ” be given .Choose “ k1, k2N” such that

(fv, fxn) <<for each “n≥k1 “ ,and

d(fv, Sxn) = d(fv, fxn+1)<<, for each” n≥k2”. Let “k = max{k1, k2}”.

Then, d(fv, Sv) << + = c .(by (4),(5) and (6)).

Since “c” is arbitrary, we get that d(fv, Sv) << for each “mN”.

Noting that → v as m →∞, we conclude that - d(fv, Sv) → 0 - d(fv, Sv) m →∞.

Since “P” is closed, then –d(fv,Sv)P. Thus d(fv,Sv) P∩(-P).

Hence, d(fv, Sv) = θ.

Therefore, “fv = Sv”. Then “f “and “S” have a coincidence point “vX”.

**3.2.Remark2.** If we choose “α4 = α5 = 0” in the above 3.1.Theorem, then we get the “Theorem 2.3” of [14].

**3.3. Remark.** If we choose “α1= λ” and “α2 = α3 = α4 = α5 = 0” in the above Theorem 2.1, then we get the following Corollary.

**3.4.Corollary .**Let (X,) be “partially ordered set” and (X, d) be a complete C-M- Space over a solid cone “P”. Let f, S: X→X be two maps such that

 d(Sx, Sy) λd(fx, fy) … (7)

for all” x, yX” ,for which “fx” and “fy” are comparable. Assume that “f” and “S” satisfy the following conditions:

(a). If {xn} is a non – increasing sequence in “X” with respect to” “such that” xn→x “

 as n→+∞, then “ xnx” for all nN.

(b). “f” is W-D- Type-(A) with respect to “S”.

(c). “fX” is complete subspace of X.

 If” λ” is a “non-negative real number” with “λ∈[0,1)”, then “f” and “S” have a coincidence point in “X”.

**Conclusion:** We obtained a “coincidence point theorem” of W- D(Weakly Decreasing) mappings of ” Type-( A) “ in CM -spaces (Cone- Metric- -Spaces), where the “cone” is not necessarily “normal” cone . Our results are more general than the results of [14].

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