**Unique Common Fixed Point Theorem for Weakly Compatible mappings of type (P) in Menger space**

**Abstract**

In this paper, we define a weakly compatible mapping of type (P) in Menger space and establish a unique common fixed point theorem for six self-mappings in this space.

**AMS Subject Classification:** 47H10, 54H25

**Keywords:** Menger space, Common fixed point, Compatible mappings, Compatible mappings of type (P).

**1.Introduction:** “One of the most significant generalization of metric space was firstly introduced by Karl Menger in 1942 called statistical metric space” [11], “often known as probabilistic metric space after 1964. Recently, in 2021, A.K. Chaudhary, K. Jha, K.B. Manandhar, and P.P. Murthy” [2] “introduced a new notion of compatible mapping of type (P) in Menger space and continuing this space study on weakly compatible by” [8], [5], [17], and [16]. The purpose of this paper is to define a new notion of weakly compatible mapping of type (p) in Menger space and establish a common fixed point theorem.

**2. Preliminaries:**

**Definition 2.1.**[3] If a function F: is

(i) is non-decreasing,

(ii) is left continuous, and

(iii) F(x) = 0 and F(x) = 1.

Then, it is said to be distribution function.

**Definition 2.2.**[3]A pair (K, F) is said to be **Probabilistic Metric Space** if the distribution function F(p, q) or , , satisfies the following conditions:

1. ,
2. and
3. &

, .

Here, F(p, q)(x) denotes the value of F(p, q) at x ∈ R.

**Definition 2.3.**[2] Two mappings are said to be Compatible mappings of type (P) in Menger space (K, F, t) iff

Whenever is a sequence in K such that for some k in K.

**Definition 2.4.**[10] Two mappings are said to be Weakly Compatible Mappings of type (P) in Menger space (K, F, t) if and only if

Whenever the sequence is in K such that for k K.

**Lemma 2.1.** [5] Let (K, F, t) be a Menger space. If there exist such that for all , , then p = q.

For proving our main result we use some basic definitions, Theorems, Propositions and Lemmas which are given in [1].

**Main Result**

Let (X, M, t) be a complete Menger Space with t (x, y) = min{x, y} for all and be mappings such that

1 A(X) VT(X) and B(X) DS(X)

2 (A, DS) and (B, VT) are weakly compatible.

3 One of A, B, D, S, V, T be continuous.

4 (B, DS) and (A, VT) are commute each other.

5 There exist a constant such that

For all x, y X, (0, 2) and q > 0 where : [0, 1] [0, 1] satisfy

(i) is continuous and non-decreasing on [0, 1]

(ii)for all n in [0, 1]

Noting that if , class of all mappings then , and for all n in [0, 1].

Then A, B, D, S, V and T have a unique common fixed point in X.

**Proof:** Consider X. Since A(X) VT(X), so there exist a point in X such that . again, since B(X) DS(X), so for , we may choose in X such that B and so on

And inductively, we take sequences and in X such that

 and , for n 0,1,2…..

Putting and for all q>0 and r = 1-p with p (0,1) in (5), we get

Or

As p 1, we obtain

 by property of

Hence we get,

Similarly, we obtain

Therefore, for every n N,

So, by lemma, is a Cauchy sequence in K.

Since the Menger space (X, M, t) is complete, so converges to a point z in X and

consequently the sub-sequences , , and of also

converges to z.

Now suppose that VT is continuous then since B and VT are weakly compatible mappings of

type (p) then by proposition, as . Putting

and in (5), we get

As n

Letting r = 1- p with p (0, 1)

 , by property of

Which implies z = VTz by lemma 2.1.

Similarly , replacing x by and y by z in (5) we have

Letting n

, as p 1

So that,

Or

 , by property of

Which implies z = Bz by lemma 2.1.

Since by , point w in X such that Bz = DSw = z.

By putting x = w and y = z in (5), we get

Therefore ,

Or, , by property of

Which implies Aw = z, by lemma 2.1.

Again, since A and DS are weakly compatible mappings of type(p) and Aw = DSw = z, by proposition,

We have for every > 0

Hence Aw = AAw = DS(DSw) = DSw

Finally, by relation (5) with x = z, y = Bz = z, we have

Or by property of

Which implies by lemma 2.1.

Hence Az = Bz = VTz = z.

Now by putting x = z, y = DSz in relation (5), we have

Or

 by property of

Which implies DSz = z by lemma 2.1.

Now to prove Sz = z, put x = Sz and y = z in relation (5)

 by property of

Which implies Sz = z by lemma 2.1.

Since DSz = z implies that Dz = z.

Now to prove Tz = z, put x = z, y = Tz in relation (5)

We have,

Or

 by property of

Therefore, Tz = z, by lemma 2.1.

Since VTz = z implies that Vz = z.

Hence .

Therefore z is common fixed point of A, B, D, S, V and T.

**Uniqueness:** suppose h is another point in X such that

.

Then putting x = z, y = h and r = 1 in relation (5), we get

 by property of

Which implies that z = h by lemma 2.1.

Hence, and z is a unique common fixed point for A, B, D, S, V and T in X.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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