Study on Coincidence Points of Weakly Decreasing Mappings of Type (A) in CM- Spaces

Abstract

In this paper, we prove a result on a coincidence point theorem of weakly decreasing maps of type A in CM -spaces (Cone Metric spaces), where the cone is not necessarily normal. These results extend and improve some well known recent results existing in the literature.

Key words: Coincidence point, cone metric space, ordered sets, weakly decreasing maps of type (A).

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1. Introduction and Preliminaries

Huang and Zhang [4] introduced the concept of a cone metric space, where the set of real numbers are replaced by an ordered Banach space and obtained fixed point theorems of contractive type mappings in a CM-space(Cone Metric space). Later on, many authors generalized these fixed point theorems to different types of contractive conditions in cone metric spaces (see, e.g.[2,5,11-13,15]). Ran and Reuring [10] proved existence of fixed point theorems in a partially ordered sets. Since then many authors have generalized their fixed point theorems in different ways (See eg. [9]). Altun et al. [3] introduced the concept of weakly increasing maps and proved some fixed point theorems. G.Jungck [6] introduced the concept of the notion of compatible maps in metric space, while G. jungck[7] introduced the notion of weakly compatible maps in metric spaces. In 2010 S. Janković et al. [8] extended the notion of compatible and weakly compatible maps in cone metric space. In 2012, W. Shatanawi [14] introduce the concept of weakly decreasing maps type (A) and obtained some coincidence point results in cone metric spaces without assuming the normal cone. The result in this paper is an extension of the results 0f [14].

Definition 1.1. [4] Let B be a real Banach space and P a subset of B .The set P is called a cone

if and only if:

(a). P is closed, non–empty and $P \neq \{\theta\}$;

(b). $a,b \in \mathbb{R}$, $a,b \geq 0$, $x, y \in P$ implies $ax+by \in P$;

(c). $x \in P$ and $-x \in P$ implies $x = \theta$.

Definition 1.2. [4] Let P be a cone in a Banach space B, define partial ordering '$\preccurlyeq$' with respect to P by $x \preccurlyeq y$ if and only if $y-x \in P$ .We shall write $x \prec y$ to indicate $x \preccurlyeq y$ but $x \neq y$ while $x \ll y$ will stand for $y-x \in \text{int } P$, where Int P denotes the interior of the set P. This cone P is called an order cone. It can be easily shown that $\lambda\ \text{int}(P) \subseteq \text{int}(P)$ for all $\lambda \in \mathbb{R}^+$.

Definition 1.3. [4] Let B be a Banach space and $P \subset B$ be an order cone .The order cone P is called normal if there exists K>0 such that for all $x,y \in B$,

$$\theta \preccurlyeq x \preccurlyeq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. [4] Let X be a nonempty set of B .Suppose that the map
$d: X \times X \to B$ satisfies :

(d1). $\theta \prec d(x, y)$ for all $x, y \in X$ and
$d(x, y) = \theta$ if and only if $x = y$;

(d2).$d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3).$d(x, y \preccurlyeq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a CM-space(Cone Metric–space).

Definition 1.5. [4]Let (X,d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c \gg \theta$,there is N such that for all n,m>N, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $c \gg \theta$,there is an N such that for all n>N, $d(x_n, x) \ll c$, for some fixed x in X .We denote this $x_n \to x$ (as $n \to \infty$) .

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.6. [14] Let $(X,\sqsubseteq)$ be partially ordered set and let f, T: $X \to X$ be two maps. We say thatf is weakly decreasing type A with respect to T if the following conditions hold:
(i). For all $x \in X$, we have that $fx \sqsubseteq fy$ for all $y \in T^{-1}(fx)$.
(ii). $TX \subseteq fX$.

Definition 1.7. [8] Let (X, d) be a C-M-space and f, g: $X \to X$ be two self-maps. The pair {f, g} is said to be compatible if, for an arbitrary sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$, and for arbitrary $c \in \text{int}(P)$, there exists $n_0 \in \mathbb{N}$ such that $d(fgx_n, gfx_n) \ll c$ whenever $n > n_0$. It is said to be weakly compatible if fx = gx implies fgx = gfx.

Definition 1.8. [1] For the mapping f, g: $X \to X$. If w = fz = gz for some z in X, then z is called a coincidence point of f and g and w is called a point of coincidence of f and g..

2. The Main Results

In this section, we prove a coincidence point theorem of weakly decreasing maps of type A in cone metric spaces, where the cone is not necessarily normal.

Theorem 2.1. Let $(X,\sqsubseteq)$ be partially ordered set and (X, d) be a CM- space over a solid cone P. Let f, T: $X\to X$ be two maps such that

$$d(Tx, Ty)\preccurlyeq\alpha_1 d(fx, fy)+\alpha_2 d(fx, Tx)+\alpha_3 d(fy, Ty)+\alpha_4 d(fx, Ty)+\alpha_5 d(fy, Tx) \quad \ldots \quad (1)$$

for all $x, y\in X$ for which fx and fy are comparable. Assume that T and f satisfy the following conditions:

(i). If $\{x_n\}$ is a non increasing sequence in X with respect to $\sqsubseteq$ such that $x_n\to x$ as $n\to+\infty$, then $x_n\sqsupseteq x$ for all $n\in N$.

(ii). f is weakly decreasing type A with respect to T.

(iii). fX is complete subspace of X.

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non-negative real numbers with $\alpha_1+ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\in[0,1)$ then T and f have a coincidence point in X, that is there exists a point $u\in X$ such that Tu = fu.

Proof: Let $x_0 \in X$. Since $TX\subseteq fX$, we can choose $x_1 \in X$ such that $Tx_0 = fx_1$. Also since $TX\subseteq fX$ We can choose $x_2 \in X$ such that $Tx_1 = f x_2$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $Tx_n= fx_{n+1}$. Since $x_n\in T^{-1}(f x_{n+1})$, $n\in N$, then by using the assumption that f is weakly decreasing of type A with respect to T, we have
$fx_0 \sqsupseteq fx_1 \sqsupseteq fx_2 \sqsupseteq\ldots$
By the condition (1) we have,

$$\begin{aligned}
d(Tx_n, Tx_{n+1})&\preccurlyeq\alpha_1 d(fx_n, fx_{n+1})+ \alpha_2 d(fx_n, Tx_n)+ \alpha_3 d(fx_{n+1}, Tx_{n+1})+ \alpha_4 d(fx_n, Tx_{n+1}) + \alpha_5 d(fx_{n+1}, Tx_n)\\
&\preccurlyeq \alpha_1 d(Tx_{n-1}, Tx_n)+ \alpha_2 d(Tx_{n-1}, Tx_n)+ \alpha_3 d(Tx_n, Tx_{n+1})+ \alpha_4 d(Tx_{n-1}, Tx_{n+1}) + \alpha_5 d(Tx_n, Tx_n)\\
&\preccurlyeq \alpha_1 d(Tx_{n-1}, Tx_n)+ \alpha_2 d(Tx_{n-1}, Tx_n)+ \alpha_3 d(Tx_n, Tx_{n+1})+ \alpha_4 [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]\\
&\preccurlyeq(\alpha_1 + \alpha_2+ \alpha_4) d(Tx_{n-1}, Tx_n)+(\alpha_3+ \alpha_4)d(Tx_n, Tx_{n+1}).
\end{aligned}$$

And

$$\begin{aligned}
d(Tx_{n+1}, Tx_n)&\preccurlyeq\alpha_1 d(fx_{n+1}, fx_n)+ \alpha_2 d(fx_{n+1}, Tx_{n+1})+ \alpha_3 d(fx_n, Tx_n)+ \alpha_4 d(fx_{n+1}, Tx_n) + \alpha_5 d(fx_n, Tx_{n+1})\\
&\preccurlyeq\alpha_1 d(Tx_n, Tx_{n-1})+ \alpha_2 d(Tx_n, Tx_{n+1})+ \alpha_3 d(Tx_{n-1}, Tx_n)+ \alpha_4 d(Tx_n, Tx_n) + \alpha_5 d(Tx_{n-1}, Tx_{n+1})
\end{aligned}$$

$$\preccurlyeq \alpha_1 d(Tx_{n-1}, Tx_n) + \alpha_2 d(Tx_{n-1}, Tx_n) + \alpha_3 d(Tx_n, Tx_{n+1}) + \alpha_5 [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$$

$$\preccurlyeq (\alpha_1 + \alpha_2 + \alpha_5)\, d(Tx_{n-1}, Tx_n) + (\alpha_3 + \alpha_5) d(Tx_n, Tx_{n+1})$$

Hence,

$$2d(Tx_{n+1}, Tx_n) = d(Tx_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1})$$

$$\preccurlyeq (\alpha_1 + \alpha_3 + \alpha_5)\, d(Tx_n, Tx_{n-1}) + (\alpha_2 + \alpha_5)\, d(Tx_n, Tx_{n+1}) + (\alpha_1 + \alpha_2 + \alpha_4)\, d(Tx_{n-1}, Tx_n) + (\alpha_3 + \alpha_4) d(Tx_n, Tx_{n+1}).$$

$$\preccurlyeq (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, d(Tx_{n-1}, Tx_n) + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) d(Tx_n, Tx_{n+1}).$$

$$(2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) d(Tx_{n+1}, Tx_n) \preccurlyeq (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, d(Tx_{n-1}, Tx_n)$$

$$\preccurlyeq \left(\frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5}\right) d(Tx_{n-1}, Tx_n).$$

Putting, $k = \left(\frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{2 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5}\right)$. We obtain,

$$d(Tx_n, Tx_{n+1}) \preccurlyeq k\, d(Tx_{n-1}, Tx_n). \qquad \ldots \qquad (2)$$

Thus, for $n \in N$, we have

$$d(Tx_n, Tx_{n+1}) \preccurlyeq k\, d(Tx_{n-1}, Tx_n) \preccurlyeq k^2\, d(Tx_{n-2}, Tx_{n-1}) \preccurlyeq \cdots \preccurlyeq k^n\, d(Tx_0, Tx_1).$$

Let $n, m \in N$ with $m > n$. Then

$$d(Tx_n, Tx_m) \preccurlyeq \sum_{i=n}^{m-1} d(Tx_i, Tx_{i+1})$$

$$\preccurlyeq \sum_{i=n}^{m-1} k^i\, d(Tx_0, Tx_1).$$

Since, $k \in [0,1)$, we have

$$d(Tx_n, Tx_m) \preccurlyeq \frac{-k^n}{1-k} d(Tx_0, Tx_1) \to \theta \text{ as } n \to \infty. \qquad \ldots \qquad (3)$$

We shall show that $\{Tx_n\}$ is a Cauchy sequence in (X, d). For this, let $c \gg \theta$ be given.

Since, $c \in int(P)$, then there exists a neighborhood of θ, $N_\delta(\theta) = \{y \in E: \| y \| < \delta\}, \delta > 0$, such that

$c + N_\delta(\theta) \subseteq int(P)$. Choose a natural number N_1 such that $\left\| \frac{-k^{N_1}}{1-k} d(Tx_0, Tx_1) \right\| < \delta$.

Then for all $n \geq N_1$ we have that $\frac{-k^n}{1-k} d(Tx_0, Tx_1) \in N_\delta(\theta)$.

Hence, $c-\frac{k^n}{1-k}d(Tx_0, Tx_1) \in c + N_\delta(\theta) \subseteq int(P)$.

Thus, we have that for all $n\geq N_1$, $\frac{k^n}{1-k}d(Tx_0, Tx_1) \ll c$. … (4)

By (3) and (4), it follows that $d(Tx_n, Tx_m) \ll c$ whenever $n\geq N_1$.

Hence, $\{Tx_n\}$ is a Cauchy sequence in X.
Since, $TX\subseteq fX$.
Therefore, $\{fx_n\}$ is a Cauchy sequence in fX. Since, fX is complete, then there exists u = fv for

some $v\in X$ such that $\lim_{n\to\infty} f\,x_n = u = fv$.

Since $\{fx_n\}$ is a non–increasing sequence in X, then
$fx_n \sqsupseteq fv$ for all $n\in N$, then by (1) we have

$$d(Tx_n,Tv)\preccurlyeq\alpha_1 d(fv, fx_n)+\alpha_2\, d(fx_n, Tx_n)+ \alpha_3 d(fv, Tv)+\alpha_4 d(fx_n, Tv)+\alpha_5 d(fv, Tx_n). \quad \ldots \quad (5)$$

By the triangle inequality and (5) we have

$$d(fv, Tv) \preccurlyeq d(fv, fx_n)+ d(fx_n, Tx_n)+ d(Tx_n, Tv)$$

$$\preccurlyeq d(fv, fx_n)+ d(fx_n, Tx_n)+ \alpha_1 d(fv, fx_n)+\alpha_2\, d(fx_n, Tx_n)+ \alpha_3 d(fv, Tv)+\alpha_4 d(fx_n, Tv) +\alpha_5 d(fv, Tx_n)$$

$$\preccurlyeq d(fv, fx_n)+ d(fx_n, Tx_n)+ \alpha_1 d(fv, fx_n)+\alpha_2\, d(fx_n, Tx_n)+ \alpha_3 d(fv, Tv)+\alpha_4 d(fx_n, Tv) +\alpha_5\,[d(fv,fx_n)+(fx_n ,Tx_n)]$$

$$\preccurlyeq(1+\alpha_1+\alpha_5)\, d(fv, fx_n)+(1+\alpha_2 +\alpha_5)\, d(fx_n, Tx_n)+ \alpha_3 d(fv, Tv)+\alpha_4 d(fx_n, Tv)$$

$$\preccurlyeq(1+\alpha_1+\alpha_5)\, d(fv, fx_n)+(1+\alpha_2 +\alpha_5)[d(fx_n,fv)+d(fv,Tx_n)]+ \alpha_3 d(fv, Tv) +\alpha_4[d(fx_n, fv)+d(fv, Tv)]$$

$$\preccurlyeq(2+\alpha_1+\alpha_2+\alpha_4+2\alpha_5)d(fv, fx_n)+(1+\alpha_2 +\alpha_5)d(fv,Tx_n)+(\alpha_3+\alpha_4)d(fv, Tv).$$

Hence, we have

$$1-(\alpha_3+ \alpha_4)\, d(fv, Tv) \preccurlyeq (2+\alpha_1+\alpha_2 + \alpha_4 +2\alpha_5)d(fv, fx_n)+ (1+\alpha_2 +\alpha_5)\, d(fv, Tx_n).$$

$$d(fv, Tv) \preccurlyeq \frac{2+\alpha_1+\alpha_2+\alpha_{4+2\alpha_5}}{1-(\alpha_3+\alpha_4)}(fv, fx_n)+ \frac{1+\alpha_2+\alpha_5}{1-(\alpha_3+\alpha_4)}\, d(fv, Tx_n). \quad \ldots \quad (6)$$

Let $c\gg\theta$ be given .Choose $k_1, k_2\in N$ such that (fv, fx_n)<<$\frac{1-(\alpha_3+\alpha_4)c}{2(2+\ \alpha_1+\alpha_2+\alpha_4+2\alpha_5)}$ for each n≥k_1, and

d(fv, Tx_n) = d(fv, fx_{n+1})<< $\frac{1-(\alpha_3+\alpha_4)c}{2(1+\alpha_2\ +\alpha_5)}$, for each n≥$k_2$. Let k = max{$k_1$, k_2}.

Then, d(fv, Tv)<<$\frac{c}{2}$ + $\frac{c}{2}$ = c .(by (4),(5) and (6)).
Since c is arbitrary, we get that d(fv, Tv)<<$\frac{c}{m}$ for each m∈N.
Noting that $\frac{c}{m}$ → v as m →∞, we conclude that $\frac{c}{m}$- d(fv, Tv) → 0 - d(fv, Tv) m →∞.
Since P is closed, then –d(fv,Tv)∈P.Thus d(fv,Tv) ∈ P∩(-P).
Hence, d(fv,Tv) = θ.
Therefore, fv = Tv. Then f and T have a coincidence point v∈X.

Remark 2. 2. If we choose α_4 = α_5 = 0 in the above Theorem 2.1, then we get the Theorem 2.3 of [14].

Remark 2.3. If we choose α_1= λ and α_2 = α_3 = α_4 = α_5 = 0 in the above Theorem 2.1, then we get the following Corollary.

Corollary 2.4. Let (X,⊑) be partially ordered set and (X, d) be a complete CM space over a solid cone P. Let f, T: X→X be two maps such that

$$d(Tx, Ty) \preccurlyeq \lambda d(fx, fy) \qquad \ldots \qquad (7)$$

for all x, y∈X for which fx and fy are comparable. Assume that f and T satisfy the following conditions:

(i). If {x_n} is a non – increasing sequence in X with respect to ⊑ such that x_n→x as n→+∞, then x_n⊒x for all n∈N.
(ii). f is weakly decreasing type A with respect to T.
(ii). fX is complete subspace of X.

If λ is a non-negative real number with λ∈[0,1), then f and T have a coincidence point in X.

Conclusion: Our results are more general than the results of [14].

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