**Unique Common Fixed Point Theorem for Weakly Compatible mappings of type (P) in Menger space**

**Abstract**

In this paper, we define a weakly compatible mapping of type (P) in Menger space and establish a unique common fixed point theorem for six self-mappings in this space.

**AMS Subject Classification:** 47H10, 54H25

**Keywords:** Menger space, Common fixed point, Compatible mappings, Compatible mappings of type (P).

**1. Introduction:** one of the most significant generalization of metric space was introduced by Karl Menger in 1942 called statistical metric space [11], often known as probabilistic metric space after 1964. The concept of a probabilistic space applies to circumstance in which we do not precisely know the distance between two points but only the probabilities of different values for this distance. In note [11], Menger outlined how to replace the numerical distance between two points x and y by a distribution function F(x, y) whose value F(x, y)(t) at a real number t is interpreted as the probability that the distance between x and y is less then t. Due to B. Schweizer and A. Skalar [3], [4] in 1960, the study of this domain was significantly broadened. This space becomes very active when V.M. Sehgal and A. T. Barucha Reid [18] 1972, obtained a contraction mapping in Menger Probabilistic metric space as a generalization of S. Banach’s[14] well known Banach Contraction Principle in metric space and developed fixed point theorems. In the study of Menger space, S. N. Mishra [15] 1991 developed a compatible mapping in the probabilistic metric space, and then many researchers worked on a large number of compatible mappings. Recently, in 2021, A.K. Chaudhary, K. Jha, K.B. Manandhar, and P.P. Murthy [2] introduced a new notion of compatible mapping of type (P) in Menger space and established a common fixed point theorem by using compatible mapping of type (P) in Menger space which is earlier introduced in metric space by H.K. Pathak, Y.J. Cho, S.S. Chang and S.M. Kang [10] in 1996. And continuing this space study on weakly compatible by [8], [5], [17], and [16]. The purpose of this paper is to define a new notion of weakly compatible mapping of type (p) in Menger space and establish a common fixed point theorem.

**2. Preliminaries:**

**Definition 2.1.**[3] If a function F: is

(i) is non-decreasing,

(ii) is left continuous, and

(iii) F(x) = 0 and F(x) = 1.

Then, it is said to be distribution function.

**Example 1**. Let H(x) stands for the heavy side function, which is defined as:

**Definition 2.2.**[3] Let (the set of all distribution functions) be a distribution function and K be a non-empty set. Then a pair (K, F) is said to be **Probabilistic Metric Space** (briefly, PM-Space) if the distribution function F(p, q), , also denoted by F(p, q) or by Fp,q satisfies following conditions:

1. ,
2. and
3. For every and for every

x, y > 0, , .

Here, F(p, q)(x) represents the value of F(p, q) at x ∈ R.

**Definition. 2.3** [13] A function T : [0, 1] [0,1] [0, 1] is referred to as triangular norm (shortly T-norm) if it satisfies the following conditions;

1. T(0, 0) = 0 and T(a, 1) = a for every a [0, 1],
2. T(a, b) = T(b, a) for every a, b [0, 1],
3. T(a, b) T(c, d) whenever a c and b d, and
4. T(a, T(b, c)) = T(T(a, b), c)) , for every a, b, c ∈ [0, 1]).

**Definition.2.4** [4] Menger Space, also known as Menger Probabilistic Metric Space, is a triplet (K, F, T), where (K, F) is a PM space, T is a T− norm and also satisfying following conditions:

, for all and .

**Definition.2.5** [19] A mapping in Menger space (K, F, t) is said to be continuous at a point if for every and , there exist and such that if

, then .

**Definition.2.6** [19] Let (K, F, t) be a Menger space and t be a continuous t-norm then,

(a) A sequence in K is said to be **Converge** to a point k in K (written ) iff for every and , there exist an integer N = (N, ) > 0 such that for all n N. in this case, we write .

(b) A sequence in K is said to be **Cauchy sequence** if for every and there exist an integer N = (N, ) > 0 such that for all n , m N.

(c) A Menger space (K, F, T) is said to be **Complete** if every Cauchy sequence in K converges to a point in K.

**Definition 2.7. [7]** Let K be a non-empty set and Q, R : K → K be arbitrary mappings, then k ∈ K is said to be a common fixed point of Q and R if Q(k) = R(k) = k.

**Example 2.** Let be functions such that and , the x = 0 is a common fixed point.

The notion of compatible mapping in Menger space was first introduced by S. N. Mishra[11] in 1991as an extension of compatible mapping in metric space introduced by G. Jungck [7] in 1986.

**Definition 2.8.** [8] Two mappings are said to be Compatible Mappings in Menger space (K, F, t) iff

 for all x > 0

Whenever is a sequence in K such that for some k in K.

The weakly commuting mappings were introduced by G. Jungck in 1996 as:

**Definition 2.9.** [8] Two mappings are said to be weakly commuting in Menger space (K, F, t) iff for all k in K and x > 0.

**Definition 2.10.**[5] Two mappings are said to be weakly compatible or coincidentally commuting in Menger space (K, F, t) if they commute at their coincidence points i.e. if then .

In 2021, A.K. Chaudhary, K. Jha, K. B. Manandhar, and P.P. Murthy [6] have introduced the following compatible mapping of type (P) in Menger space as an extension of H.K. Pathak et.al [12] as follows:

**Definition 2.11.**[2] Two mappings are said to be Compatible mappings of type (P) in Menger space (K, F, t) iff

Whenever is a sequence in K such that for some k in K.

Now we introduce weakly Compatible mappings of type (P) in Menger space with an example as follows:

**Definition 2.12.**[10] Two mappings are said to be Weakly Compatible Mappings of type (P) in Menger space (K, F, t) if and only if

Whenever is a sequence in K such that for some k in K.

**Example 3**[1]. Let (K, d) be a metric space where K = [0,2] with usual metrics d(x, y) = |x-y| and let (K, F) be PM-space with

For all . Let be defined by

And

Taking sequence where . Then , . Also and . So that for all t > 0 and for all t > 0.

Therefore we have for all x > 0. Hence (Q, R) are weakly compatible mappings of type (P) but it is neither compatible mappings of type (P) nor compatible mappings.

**Theorem 2.1.**[19] Let (K, F, t) be a Menger space with continuous t- norm and be self mapping. Then Q is continuous at a point k K if and only if for every sequence in K converging to a point k then sequence converges to the point Qk. i.e. then it implies .

**Proposition 2.1.** [10] In Menger space (K, F, t), if for all then t(a, b) = min (a, b) for all a, b [0, 1].

**Lemma 2.1.** [5] Let (K, F, t) be a Menger space. If there exist such that for all , , then p = q.

**Proposition 2.2.** [1]Let (K, F, t) be a Menger space such that T – norm is continuous and for all and is continuous mappings. Then, Q and R also written as (Q, R), are weakly compatible mappings of type (P) if they are compatible mappings of type (P).

Proof. Suppose Q and R be compatible mappings of type (P). Then, we have, . So, (Q, R) be weakly compatible mappings of type (P).

**Proposition 2.3.**[1] Let (K, F, t) be a Menger space such that the T – norm t is continuous and for all and is continuous mappings. Then, Q and R also written as (Q, R), are compatible mappings of type (P) if they are weakly compatible mappings of type (P).

Proof. Let be a sequence in K and since Q and R be continuous mappings. Then by theorem 2.1, we have for some k in K. If Q and R are weakly compatible mappings of type (P). Then, we have , for all x > 0. So, (Q, R) be compatible mappings type (P).

**Proposition 2.4.** Let (K, F, t) be a Menger space such that T – norm t is continuous and for all and be mappings. If Q and R are weakly compatible mappings of type (P) and Qk = Rk for some then, .

Proof. Suppose is a sequence in K defined by where k = 1, 2, 3,….. for some and Qk = Rk. Then we have as . Since Q and R are weakly compatible mappings of type (P), then for every , . So, QQk = RRk, since Qk = Rk implies QQk = QRk = RQk = RRk.

**Proposition 2.5.** [1] Let (K, F, t) be a Manger space such that the T−norm t is continuous and t(x, x)x for all x ∈ [0, 1] and Q,R : K → K be mappings. Let Q and R be weakly compatible mappings of type (P) and  for some k in K. Then we have,

(i) if Q is continuous at k,

(ii) if R is continuous at k,

(iii) QRk = RQk and Qk = Rk if Q and R are continuous at k.

Proof. (i) Suppose that Q is continuous at k. Since, we have for some k in K. So, , as lim n. Again, since Q and R are weakly compatible of type (P), so for every . Therefore, we have

by definition of Menger space or,

n

This implies that . So, .

(ii) we may prove (ii), as we prove (i)

(iii) suppose that Q, R : K K is continuous at k. so, by (i), , as . On the other hand, since , as and R is continuous at k. so, by proposition 2.5 (ii), we get, . Thus, we have Qk = Rk by the uniqueness of the limit and so by preposition 2.4, we get QRk = RQk. Hence proved.

The following lemma needs to prove the main theorem:

**Lemma 2.2.** [5] Let be a sequence in Menger space (K, F, t), where t is continuous T−norm and for all x ∈ [0, 1]. If there exists a constant k ∈ [0, 1] such that , for all x > 0 and n ∈ N, then is a Cauchy sequence in K.

**Main Result**

Let (X, M, t) be a complete Menger Space with t (x, y) = min{x, y} for all and be mappings such that

1 A(X) VT(X) and B(X) DS(X)

2 (A, DS) and (B, VT) are weakly compatible.

3 One of A, B, D, S, V, T be continuous.

4 (B, DS) and (A, VT) are commute each other.

5 There exist a constant such that

For all x, y X, (0, 2) and q > 0 where : [0, 1] [0, 1] satisfy

(i) is continuous and non-decreasing on [0, 1]

(ii)for all n in [0, 1]

Noting that if , class of all mappings then , and for all n in [0, 1].

Then A, B, D, S, V and T have a unique common fixed point in X.

**Proof:** Consider X. Since A(X) VT(X), so there exist a point in X such that . again, since B(X) DS(X), so for , we may choose in X such that B and so on

And inductively, we may construct sequence and in X such that

 and , for n 0,1,2…..

Putting and for all q>0 and r = 1-p with p (0,1) in (5), we get

Or

As p 1, we obtain

 by property of

Hence we get,

Similarly, we obtain

Therefore, for every n N,

So, using lemma (2.2), is a Cauchy sequence in K.

Since the Menger space (X, M, t) is complete, so converges to a point z in X and

consequently the sub-sequences , , and of also

converges to z.

Now suppose that VT is continuous then since B and VT are weakly compatible mappings of

type (p) then by proposition (2.5), as . Putting

and in relation (5), we get

As n

Letting r = 1- p with p (0, 1)

 , by property of

Which implies z = VTz by lemma 2.1.

Similarly , replacing x by and y by z in relation (5) we have

Letting n

, as p 1

So that,

Or

 , by property of

Which implies z = Bz by lemma 2.1.

Since by , there exist a point w in X such that Bz = DSw = z.

By putting x = w and y = z in relation (5), we have

Therefore ,

Or, , by property of

Which implies Aw = z, by lemma 2.1.

Again, since A and DS are weakly compatible are weakly compatible mappings of type(p) and Aw = DSw = z, by proposition 2.4,

We have for every > 0

Hence Aw = AAw = DS(DSw) = DSw

Finally, by relation (5) with x = z, y = Bz = z, we have

Or by property of

Which implies by lemma 2.1.

Hence Az = Bz = VTz = z.

Now by putting x = z, y = DSz in relation (5), we have

Or

 by property of

Which implies DSz = z by lemma 2.1.

Now to prove Sz = z, put x = Sz and y = z in relation (5)

 by property of

Which implies Sz = z by lemma 2.1.

Since DSz = z implies that Dz = z.

Now to prove Tz = z, put x = z, y = Tz in relation (5)

We have,

Or

 by property of

Therefore, Tz = z, by lemma 2.1.

Since VTz = z implies that Vz = z.

Hence .

Therefore z is common fixed point of A, B, D, S, V and T.

**Uniqueness:** suppose h is another point in X such that

.

Then putting x = z, y = h and r = 1 in relation (5), we get

 by property of

Which implies that z = h by lemma 2.1.

Hence, and z is a unique common fixed point for A, B, D, S, V and T in X.

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