STOCHASTIC ALMOST SURE EXPONENTIAL SELF STABILIZATION OF NON-LINEAR OPTIMAL CONTROL DELAY INTEGRO - DIFFERENTIAL EQUATIONS DRIVEN BY ITO-TYPE NOISE .

Abstract

This study explores the application of multiplicative Ito-type noise in stabilizing nonlinear optimal control delay differential equations $(OCDDE_s)$ that are generally unstable in their deterministic form . The equation is perturbed by a multiplicative Ito- type noise to form a stochastic optimal control delay differential equation .The noise scaling parameter in the comparable stochastic optimal control system is replaced with finite integral expression by making it sufficiently as large as possible to stochastically self stabilized the resulting stochastic system in an almost sure exponential sense , under additional conditions and sufficiently small time lag . This phenomenon does not occur in deterministic

optimal control delay differential equations where noise is absent , since its solutions still admit instability .

Keywords : Almost sure exponential stability , Optimal control , deterministic delay differential equation , Lyapunov sample exponent , Brownian noise , stochastic self stabilization

1.Introductions

Stochastic delay differential equation have found many applications in science and technology such as physics, chemistry, structural systems like mechanics , optical bi- stability , fatigue cracking , financial mathematics, mathematical biology, radio astronomy , turbulent diffusion etc.

A stochastic differential equation (SDEs) is an equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process . Ito 1994 laid the foundation of a stochastic calculus known today as the Ito calculus. This represents the stochastic generalization of the classical differential calculus, which models various phenomena in continuous time such as the dynamics of stock prices, physical systems or motion of a microscopic particle subjected to random fluctuations. The corresponding stochastic differential equations (SDEs) with retarding argument generalize the ordinary deterministic differential delay equation (ODDEs) when subjected to environmental disturbances . Stochastic delay equations (SDE_S) are ordinary differential equations perturbed by noise .

In general, consider the first order nonlinear ordinary delay differential equation has the form

$$\left.\begin{aligned} \acute{x}(t) &= f(x(t), X(t-\sigma), t)dt, t \geq 0 \\ x(t) &= \varphi(t) \quad , t \in [-\Gamma, 0] \end{aligned}\right\} \quad 1.1$$

Equation (1.1) can be perturbed by noise to become

$$\left.\begin{aligned} dX(t) &= f(X(t), X(t-\sigma), t)dt + \mu g\big(X(t)\big)dB(t), t \geq 0 \\ x(t) &= x(t_0) \quad , t \in [-\Gamma, 0] \end{aligned}\right\} \quad 1.2$$

where (1.1) and (1.2) have the same initial function , $f(X(t), X(t-\sigma), t)dt$ is called the drift function , $\mu g\big(X(t)\big)dB(t)$ is called the diffusion function , μ is the noise scaling parameter which measures the fast fluctuation effect of the noise and $B(t)$ is a one dimensional Brownian noise given as $B = \{B_t\}_{t\geq 0}$ defines the randomness of the physical systems and it is often called the Ito type noise. The wiener process is the simplest intrinsic noise term that adequately model Brownian motion. Eq. (1.2) can be expressed in an integral form as

$$X(t) = X(t_0) + \int_0^t f(X(s), X(s-\sigma), s)ds + \int_0^t \mu g\big(X(s)\big)dB(s), t \geq 0 \qquad 1.3$$

The first integral in (1.3) is a Volterra integral term and the second integral is an Ito stochastic integral with respect to the Brownian motion $B = \{B_t\}_{t\geq 0}$.

Definition

Let (Γ, Ω, P) be a complete probability space with filtration $\{F_t\}_{t \geq 0}$. A standard one dimensional Brownian motion is a real valued continuous F_t- adapted process which satisfies the following properties

(i) . $B_0 = 0$

(ii) . the function $t \to B(t)$, is continuous

(iii) .for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$

(iv) . Increments of Brownian motion on non overlapping intervals are independent , i.e. $(s_1, t_1) \cap (s_2, t_2) = \emptyset$. The random variables $B_{t2} - B_{s2}, B_{t1} - B_{s1}$ are independent .

(v) . Paths of Brownian motion are not differentiable functions .

Gu .et. al, (2016) examined the almost sure exponential stability of the multi-dimensional nonlinear stochastic differential delay equation (SDDE) with variable delays of the form.

$$dX(t) = \alpha(X(t-\delta), t)dt + \beta(X(t-\sigma), t)dB(t) \qquad (1.4)$$

Where $\delta, \sigma: \mathbb{R}_+ \to [0, \Gamma]$ are the delays. The corresponding deterministic delay differential equation (DDE) has the form

$$\acute{x}(t) = \alpha(x(t-\delta), t)dt \tag{1.5}$$

admits the Lyapunov function , there exists a positive number δ such that the $SDDE$ is almost surely exponentially stable , since the delays were bounded by $t \in [-\Gamma, 0]$ where $\Gamma = max\{\delta, \sigma\}$. Different types of stochastic differential equations have been used to model various phenomena in many fields , such as unstable stock prices in finance (merton , 1976) . In other words, past events explicitly influence future results. Delay differential equations are more applicable than ordinary differential equations (ODEs), in which future behavior only implicitly depends on the past. Systems of ordinary differential equations are independent on previous state or systems of differential equations that are dependent on previous states are called systems of delay differential equations (DDEs). Delay differential equations were introduced to create more realistic models since many processes depend on past history . When dynamical systems are subjected to performance criterion , aimed at achieving certain objectives , the stability of such systems can form an area of interest to research on . Lyapunov introduced the concept of stability into the study of dynamic system since 1892. Stability means insensitive of the system to small changes in its initial state or parameters of the system .

To understand stability, it is worthy to note that stability is of different types viz: stability in probability, asymptotic stability, stochastic stability , moment stability, almost sure exponential stability, mean square stability, etc. For a system to be stable , the trajectories which were close to each other at a specific instant should therefore remain close to each other at all subsequent instances (Mao 1997) .

Zhu and Huang (2020) established the p^{th} moment exponential stability problem for a class of stochastic delay nonlinear system driven by general Brownian motion. Zong ,et . al (2018) investigated the asymptotic properties of systems represented by stochastic functional differential equations .

Mao (2008) , obtained different types of stabilities for stochastic delay differential equations of the form

$$\left.\begin{array}{c} dX(t) = f\big(X(t), X(t-\tau_1)\big)dt + g\big(X(t)X(t-\tau_2)\big)dB(t)\ , t \geq 0 \\ X(t) = \varphi(t)\ ,\ \ t \in [-\Gamma, 0] \end{array}\right\}$$

where $\Gamma = max\{\tau_1, \tau_2\}$

where $\big(B(t)\big)$ is the Brownian motion ,$f(*)$ is the drift term and $g(*)$ is the diffusion

Liu (2017) established a theory about the property of almost sure path-wise exponential stability for a class of stochastic neutral functional differential equations by developing a semi group scheme for the drift part of the systems under consideration and with path-wise stability through a perturbation approach rather than moment stability. Nane and Ni (2017) studied the stabilities of stochastic differential equations (SDE_S) driven by time changed Levy noise in both probability and moment sense . Atonuje (2015) established that , by the uses of Lyapunov function and the ideas of generalized moment inequalities as well as borel - Cantelli lemma under additional conditions on the drift and diffusion functions, p^{th}- moment , exponential stability implies the almost sure exponential stability . Zhu et .al (2021) established the almost sure exponential stability and exponential stabilization of solution to time changed stochastic differential equation . Zhang et al ,(2019) studied the problem of stabilization and destabilization of nonlinear stochastic differential delay equations .The techniques applied was the Lassalle – type stability theorems ,the non - negative semi – martingale convergence theorem and the law of large numbers for martingales .Their proposed results can be applied to study the stabilization and destabilization of more general nonlinear stochastic dynamical time delay systems . Atonuje and Ezenweani (2011) investigated the stability behavior of a non-linear deterministic delay differential equation with two time lags of the form ,

$$\left.\begin{array}{c} \frac{dx(t)}{dt} = \alpha(x(t), x(t-\sigma), t) + \beta(x(t), x(t-\tau), t), t \geq 0 \\ x(t) = \phi(t) \ , t\epsilon[-ŕ, 0] \end{array}\right\} \quad (1.6)$$

where ŕ=max{σ ,τ }, σ, τ ϵ(0,∞) , σ<τ are two constant time lags ,the initial function $\phi(t)\epsilon c([ŕ,0],\Re), \alpha\ and\ \beta$ are smooth functions which satisfied the following axioms

(i) $0\leq\frac{\alpha(x)}{x}\leq\sigma\ , x\neq 0$

(ii) $0\leq\frac{\beta(x)}{x}\leq\ \tau\ , x\neq 0$.

Eq. (1.6) were stochastically perturbed by a multiplicative Ito-type white noise to form a stochastic delay differential equation of the form

$$\left.\begin{array}{c} dX(t)=\left[\alpha\big(X(t),X(t-\sigma)\big)+\beta\big(X(t),X(t-\tau)\big)\right]dt+\lambda q(X(t),t)dB(t), t\geq 0 \\ X(t)=\varphi(t) \end{array}\right\} \quad (1.7)$$

$\alpha,\beta\epsilon L^1\left([t_0,T],R^{d.m}\right), \lambda\ is\ the\ noise\ scaling\ parameters\ \ and\ \{B(t)\}_{t\geq_0}$ is an n - dimensional Brownian motion representing the multiplicative white noise Ito-type , it was established that the presence of Brownian noise could stochastically stabilized an unstable deterministic delay differential equation ($DDEs$) . Atonuje et . al (2024) investigated the roles of a multiplicative Ito - type Brownian noise to stochastically stabilized the evolution of optimal control dynamical system with a volterra functional , described by an unstable nonlinear classical delay differential equation . The authors perturbed the equation by a multiplicative Brownian noise to form a stochastic optimal control system . The noise scaling parameter in the stochastic system were replaced with a finite integral expression , the system become stochastically self stabilized in an almost sure exponential sense , under certain conditions and sufficiently small time delay . Mao (2010) obtained the strong mean square convergence theory for the numerical solutions of stochastic delay differential equations under the local lipcshitz conditions .Also , imposed two conditions to guarantee the existence and uniqueness of the original solutions. Zhu et .al (2017) presented sufficient conditions for almost sure exponential stability of solutions to time - changed stochastic differential equations . The technique involves the construction of proper Lyapunov function and generalized Lyapunov methods to time changed stochastic differential equations . In contrast , to the almost sure exponential stability in existing papers and established new results on the stability of solutions to the time - changed stochastic differential equations . Time lag was taken into account for the discrete time state and mean square exponential stability of the controlled system was presented by (Qin et . al , 2016) . Delay feedback control based on discrete time state observation for stochastic differential equations with Markovian switching was considered by (see Mao , 2013) .

Although, a lot of monographs, conference papers and journal articles have been written by various authors on the stability of dynamical systems usually represented by ordinary differential equations, delay differential equations , stochastic differential equations etc , to the best of my knowledge , the optimal control systems is almost absent from the existing literatures . It appears that a little effort have been made on the investigation of the multiplicative Ito-type noise to the stabilization and destabilization of solutions of optimal control systems .

2 **. Problem formulation and Notations**

We consider a dynamical system (X) whose state at time t is described by an n−dimensional vector $x(t) = (x_1(t), \dots, x_k(t)$. Assume that the system (X) is controlled by certain controllers ,if these controllers are characterized at time t by an $n - dimensional$ vector $r(t) = (r_1(t), r_2(t), \dots, r_n(t)$. Suppose Ω is a compact and convex set in the $n - dimensional$ space A^n with points $r = (r_1, \dots, r)$,where $r(t)$ is called the control vector or the control function , Ω is called the control region . A measurable function $r(t)$ defined for $t \in [t_0, t]$, where $\dot{t} \in [t_1, t_2]$ is called admissible, if their range is in Ω . Let Ψ be the set of all admissible control . Suppose that (X) is the states corresponding to the time interval $I = [\gamma, t_0]$,where $|\gamma|$ is sufficiently large , is described by an $n - dimensional\ vector$

$$\Omega(t) \in \varsigma([\gamma, t_0], \omega) \qquad 2.1$$

where ω is a compact region in the set A^n , containing the origin as an interior point . Given an $r - dimensional$ continuous real - valued function , the initial function φ , on the interval $\gamma \le t \le t_0$ and $\varphi(t_0)$, find a function $x(t)$, continuous for $t \ge \gamma$ such that

$$x(t) = \varphi(t)\ , for\ \gamma \le t \le t_0\ , \qquad 2.2$$

Let ϕ be the set of all allowable initial function $\varphi(t) and\ E\ be\ the$ set of all allowable control vector $r(t)$,if $\dot{\Psi}$ is a subset of the $n - dimensional$ space of points with co-ordinates $(r_1, r_2\ , . . .\ . r_n)$, suppose that the vector $r(t) \in \dot{\Psi}$. Then the set $\dot{E}$ is called the control region where $\dot{\Psi}$ is the range of all $r(t) \in \Psi$. Any element $\varphi(t) \in \varphi$ is said to be admissible initial function and any $r(t) \in \Psi$ is said to be an admissible control . A pair $\{\varphi, u\}$ with $\varphi(t_0) \in \varphi\ \ and\ r \in \Psi$ is called admissible pair or admissible policy . Let $r(t) = r(t, t_0, \varphi, v)$ be a trajectory corresponding to an admissible pair $\{\varphi(t_0), u(t)\} \in p$. suppose that .

a. the trajectories $r(t)$ which corresponding to admissible pairs $\{\varphi(t_0), u(t)\}$ remain in a given compact region $\omega \in A^n\ \ for\ r(t) = \varphi(t),\ t \in [\alpha, t_0]$.

b the control region Ψ^* is compact , convex and contains the origin of A^n as an interior point and the members of the set Ψ of all admissible controls consist of all measurable functions defined for $t \in [t_0, t]$, [$\dot{t} \in t_1, t_2$] , whose range is contained in Ψ^*

c . during the evolution of the process ,the integral constraints

$$\int_{t_0}^{t} r_i(r, v, t)dt \subseteq W_i \text{ , } i = 1,2 \text{ . . . } n \text{ .}$$

d . the region ω is compact ,convex and contains the origin of A^n as an interior point and the set φ of all admissible initial function which is defined for $\varphi = \{\varphi(t) \in c([\gamma, t_0], \omega\}$, $\varphi(t)$ is admissible , compact ,convex and contains the origin as an interior point . where w_i *is a closed subsets of* A^n

We consider the optimal controlled system (X) which is described by the nonlinear deterministic ordinary delay differential equation of the form

$$\left.\begin{array}{r} x'(t) = f(x(t), r(t), x(t-\tau\ ), t\ )\ t > 0) \\ x(t) = \varphi(t), t \in [-\Gamma, 0] \end{array}\right\} \qquad (2.3)$$

where $f(x(t), r(t), x(t-\tau), t)dt$ is a volterra functional defined and bounded for $t \in [-\Gamma, 0]$ satisfying the following conditions (1)-(4) holds for all $v(t)\epsilon\varphi$ *and* $\forall$ $x(t)\epsilon\varsigma(I[t_0, T], \omega)$, where ω is a compact subset of an n-dimensional space $\dot{E}$, F is integrable and continuous for x *and* r , $\tau\epsilon(0,1)$ is a constant time lag or delay, $r(t)$ is the control vector, defined for $t \in [-\Gamma, 0]$ and $x(t) = \varphi(t), t \in [-\Gamma, 0]$ with $\varphi(t_0) \in \varphi$ is the initial datum. By the solution of (2.3) , we mean a continuous vector function $x(t) = \varphi(t)$ such that $x(t)$ satisfies $Eq.$ (2.3) as well as the initial condition $x(t) = \varphi(t), t \in [-\Gamma, 0], \varphi(t_0) \in \varphi$. The solution of equation (2.3) is said to be unstable on [-Γ,0] , if for every $\varepsilon > 0$ and for any $t \geq 0$. Assume that $x(t\ \ t_0, x_0)$ is the solution of (2.3) with the initial datum $x(t) = \varphi(t)$, where $t \geq t_0$ and $x_0 \in \mathbb{R}$ .Then $x(t) = \varphi(t)$ is said to be unstable if for any $\varepsilon > 0$ and any $t \geq t_0$, there exists a $\xi = \xi(\varepsilon, t_0) < 0$ such that $\left| x(t_0) - \varphi(t) \right| > 0 \ \Rightarrow\ \left| x(t, t_0, x_0) - \varphi(t) \right| > \varepsilon\ \forall\ t \geq t_0$

We perturbed Eq. (2.3) by multiplicative Ito-type Brownian white noise into a stochastic optimal control delay differential equation of the form

$$\left. \begin{aligned} dX(t) &= f(X(t), r(t), X(t-\tau), t)dt + \delta h(X(t), t)dB(t), t > 0 \\ x(t) &= \varphi(t)\ , t \epsilon [-\Gamma, 0] \end{aligned} \right\} \quad (2.4)$$

where $f \epsilon L^1([t_0, T]\ , R^d)$, $h \epsilon L^2([t_0, T], R^{dn})$, $r(t)$ is the control vector , δ is the noise scaling parameter which determine the strength of the fluctuation of the system , $f(*)$ is called the drift function ,$h(*)$ is called diffusion function , τ ∈(0,1) *is called the constant time lag* and $B(t)$ is a Brownian motion , such that $B = \{B(t), t \geq 0\}$, defined on the probability triple $(\varphi, \mathcal{F}, P)$ with filtration $\{X(t)\}_{t \geq 0}$, defined on $\{\varphi, \mathcal{F}, P\}$ satisfying (2.4) together with the initial datum which is the same with that of the $OCDDE_S$ (2.3). We intends to ask this question, what can replace the noise intensity parameter in the stochastic equation such that the system would be stochastically self stabilized ?.By replacing the noise scaling parameter δ in equation (2.4) with finite integral expression

$\delta = \int_0^t |\alpha(.)x(.)|^n \, ds$ into equation (2.4) , we have

$$\left.\begin{aligned} dX(t) &= f(X(t), r(t), X(t-\tau), t) + \left(\int_0^t |\alpha(.)X(.)|^n\right) g(X(t), t)dB \\ X(t) &= \varphi(t) \;\;,\;\; t \geq 0 \end{aligned}\right\} 2.$$

Eq. (2.5) becomes stochastic optimal control integro delay differential equation ($SOCIDDE_S$) .

Definition (The trivial solution of SOCIDDE)

Assume that $\{X(t)\}_{t\geq 0}$ is the solution of (2.5) . Suppose that $f(0,0,0,t) \equiv 0, g(0,0,0) \equiv 0 \;\; \forall \, t > 0$. It follows that (2.5) has the solution $X(t) \equiv 0$ corresponding to the initial datum $X(t_0) \equiv 0$ is called the trivial solution of (2.5) .

Definition (2)

The equilibrium solution $X(t, 0,0,0)$ of the SOCIDDE (2.5) is said to be almost surely exponentially stable if

$$\lim_{t\to\infty} sup \frac{1}{t} log|X(t, t_0, x_0 , i| < 0 \;\; \forall \, X_0 \in R^d \; .$$

The following Assumptions , Lemmas and Theorems are based on the Ito formula is called exponential martingale inequality . it is useful to the proof of the main results .

Assumption(1)

H1: we assume that the following hypothesis hold

(i) $|X^T \Delta f(X,t) + X^T \Delta g(X,t) \leq H|X^T \Delta X|^2|$

(ii) Trace $(h^T(X,t)\Delta h(x,t) \leq \pi X^T \Delta X)$

(iii) $|h^T(X,t)\Delta h(x,t)|^2 \geq \gamma |X^T \Delta X|^2 \;\; \forall \;\; t > 0 \, , and \; x \in R^d$

H2: There exists a pair of constants $M > 0 \; and \; y \geq 0$ such that $\|\alpha(t)\| \leq Me^{\alpha t}$

$\forall \, t \geq 0$, where $\alpha(t)$ is convergence rate function given by $\alpha(t)e^{\alpha t Id.d}$ where $I_{d.d}$ is the $d \times d$ identity matrix

Definition (convergence rate function)

A function $\alpha(t)$ is a continuous real valued function on R^{nd} and a positive constant $M > 0$ such that $\|\alpha(t)\| \leq Me^{\alpha t}$,$\forall\, t \geq 0$.

The convergence rate function plays a vital role in making the noise scaling parameter large enough to stabilized the system almost surely exponentially self stabilized .

3. Main result

Lemma (3)

If (H1) hold . Then the solution of the stochastic optimal control integro-differential delay equation

$$dX(t) = f(X(t), V(t), X(t-\tau), t)dt + \left(\int_{t_0}^{t} |\alpha(s)X(s)|^n g(x(t), t)dB(t)\right)$$

has the property $P\{x(t, x_0) \neq 0\ \forall\, t \geq 0\} = 1$, almost surly such that $x_0 \neq 0$.

Lemma (3.1)

Consider the system

$$dX(t) = [f(X(t), V(t), X(t-\tau), t)]dt + \left(\int_{t_0}^{t} |\alpha(s)X(s)|^n g(x(t), t)dB(t)\right) \quad (3.1)$$

suppose that there exists a function $\alpha \in C^{2,1}(R^d \times [0, \infty), R)$, the family of all non negative function$\alpha\ (x, t)$ define on $R^d \times [0, \infty), R)$ such that they are continuously twice differential in $x\ and\ t\ and\ constants\ h > 0, k > 0, q_2 \in R, q_3 \geq 0$ such that $\forall\ x \neq 0\ , t \geq 0$,we have

(a) . $q_1|x|^h \leq \gamma(x, t)$

(b) .$L\gamma(x, t) \leq q_2\gamma(x, t)$

(c) . $|\gamma_x(x, t)g(x, t)|^2 \geq q_3\gamma^2(x, t)$. Then

$$\lim_{t\to\infty} sup \frac{1}{t} log|x(t, t_0, x_0)| \leq -\frac{q_3 - 2q_2}{2n} \quad (3.2)$$

Then the solution of the stochastic optimal control integro-differential delay equation (3.1) almost surely $\forall\ x_0 \in R^d$. If in particular that $q_3 > 2q_2$, the solution of stochastic optimal control delay differential equation (3.1) is almost surely exponentially self stabilized .

Theorem (4)

If (H1) and (H2) holds . Then the solution of the stochastic integro-differential delay equation (3.1) satisfies the property that

$$\int_0^\infty \mid h(t)x(t,x_0) \mid^n < \infty , \forall\, x_0 \in R \qquad (3.3)$$

Proof

Since $x(t,0) \equiv 0$ guarantees (H1) , we show that (3.3) holds for

$$x_0 \neq 0.$$

For every $x_0 \neq 0$, lemma (1), the solution $x(t,x_0) \neq 0$, with positive probability for all $t \geq 0$ almost surely . If the property (3.3) is not true By contradiction , there exists some $x_0 \neq 0$ for which $p(\boldsymbol{\psi}^*)$ ≥0 , where

$$\boldsymbol{\varphi}^* = \{\omega\epsilon\boldsymbol{\varphi}: \int_{\mathbf{0}}^{\infty} \mid \boldsymbol{h}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t},\boldsymbol{x_0}) \mid^{\boldsymbol{n}} \boldsymbol{dt} = \infty\}$$

Let $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t},\boldsymbol{x_0})$. By condition H1 and Ito formula , we show that for every $\boldsymbol{t} \geq \mathbf{0}$, we have

$$\mathbf{log}(\boldsymbol{X^T}(\boldsymbol{t})\mathrm{H}\boldsymbol{x}(\boldsymbol{t})) \leq \mathbf{log}(\boldsymbol{x^T}\mathrm{H}\,\boldsymbol{x_0}) + \frac{\mathbf{2}\boldsymbol{mt}}{\boldsymbol{\lambda_{min}}(\Omega)} + \tau \int_{\mathbf{0}}^{\boldsymbol{t}} \left(\int_{\mathbf{0}}^{\boldsymbol{s}} |\boldsymbol{h}(\boldsymbol{v})\boldsymbol{x}(\boldsymbol{v})|^{\boldsymbol{n}}\, \boldsymbol{dv}\right)^{\mathbf{2}} \boldsymbol{ds} \quad *$$

$$-\mathbf{2}\int_{\mathbf{0}}^{\boldsymbol{t}} \left(\int_{\mathbf{0}}^{\boldsymbol{s}} |\boldsymbol{h}(\boldsymbol{v})\boldsymbol{x}(\boldsymbol{v})|^{\boldsymbol{n}} \boldsymbol{dv}\right)^{\mathbf{2}} \frac{\left|\boldsymbol{X^T}(\boldsymbol{s})\Omega\boldsymbol{g}(\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{s})\right|^{\mathbf{2}}}{\left(\boldsymbol{x^T}(\boldsymbol{s})\Omega\boldsymbol{x}(\boldsymbol{s})\right)^{\mathbf{2}}}\, \boldsymbol{ds} + \boldsymbol{N}(\boldsymbol{t})$$

.

where

$$\boldsymbol{N}(\boldsymbol{t}) = \mathbf{2}\int_{\mathbf{0}}^{\boldsymbol{t}} \left(\int_{\mathbf{0}}^{\boldsymbol{s}} |\boldsymbol{h}(\boldsymbol{v})\boldsymbol{x}(\boldsymbol{v})|^{\boldsymbol{n}} \boldsymbol{dv}\right)^{\boldsymbol{n}} \frac{\boldsymbol{x^T}(\boldsymbol{s})\Omega\boldsymbol{g}(\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{s})}{\boldsymbol{x^T}(\boldsymbol{s})\Omega\boldsymbol{x}(\boldsymbol{s})} \boldsymbol{dB}(\boldsymbol{s})$$

is a continuous martingale vanishing at point $\boldsymbol{t} = \mathbf{0}$. For $\boldsymbol{C} = \mathbf{1}, \mathbf{2}, \ldots,$. and by the exponential martingale inequality , we have

$$p\left\{\partial: \sup_{0\leq t\leq c}\left[N(t) - \frac{2\gamma - \alpha}{8\gamma}\, \langle N(t), N(t)\rangle\right]\right\} > \frac{8\gamma logc}{2\gamma - \alpha} \leq \frac{1}{c^2}$$

where

$$\langle N(t),N(t)\rangle = 4\int_0^t\left(\int_0^s |h(V)x(v)|dv\right)^2 \frac{|x^T(S)\Omega g(X(S),S)|^2}{(x^T(s)\Omega x(s)^2} \, ds$$

By the Borel Cantelli lemma we see that for almost all $\partial \in \boldsymbol{\psi}$, there exists a random integer $\boldsymbol{c}(\boldsymbol{\partial})$ such that $\forall\ \boldsymbol{c} \geq \boldsymbol{c_1}$ we have

$$\sup_{0\leq \mathrm{t} \leq \mathrm{k}}\left[N(t) - \frac{2\tau-\alpha}{8\tau}\langle N(t),N(t)\rangle\right] \leq \frac{8\gamma logc}{2\gamma-\alpha} \qquad (3.4)$$

for $0 \leq t \leq c$,

$$N(t) \leq \frac{8\gamma \log C}{2\gamma-\mu} + \frac{2\gamma-\rho}{8\gamma}\langle H(t),H(t)\rangle$$

$$\leq \frac{8\gamma \log C}{2\gamma-\mu} + \frac{2\gamma-\rho}{2\gamma}\int_0^t (|h(u)x(u)|^n du)^2 \frac{|x^T(s)\eta g(x(s),s)|^2}{\left(x^T(S)\eta x(s)\right)^2} ds \qquad (3.5)$$

Put $(*)$ into (3.5) and by $(H1)$, we have

$$\log\left(x^T(t)Hx(t)\right) \leq \log(x_0 H x_0) + \frac{2mt}{\varphi_{min}(\Omega)} + \frac{8\gamma \log c}{2\gamma-\mu}$$

$$-\frac{2\gamma-\mu}{2}\int_0^t\left(\int_0^s |h(u)x(u)|^n du\right)^2 ds \qquad (3.6)$$

$\forall\ 0 \leq t \leq c\,, c \geq c_1$

, almost surely . By the definition of φ^{*} , we observed that for every $\partial \in \varphi^*$, there exists a random integer $z_2(\varphi)$ such that

$$\int_0^t |h(s)x(s)|^n ds \geq \sqrt{\frac{4k/\varphi_{min}(\Omega)+4\mu+8}{2\gamma-\mu}}\ \forall\ t \geq z_2 \qquad (3.7)$$

From (3.6) *and* (3.7) almost $\partial \in \varphi^*$, if $c-1 \leq t \leq c\,, c \geq c_1 \vee (c_2+1)$ .Then

$$\log\left(x^T(t)Hx(t)\right) \leq \log(x_0 H x_0) + \frac{2mt}{\varphi_{min}(\Omega)} + \frac{8\gamma \log c}{2\gamma-\mu}$$

$$-\frac{2\gamma-\mu}{2}\int_0^t\left(\int_0^s|h(u)x(u)|^n du\right)^2 ds \tag{3.8}$$

$$\leq \log\left(x^T(t)Hx(t)\right)+\frac{2mt}{\varphi_{min}(\Omega)}+\frac{8\gamma\log c}{2\gamma-\mu}-\left(\frac{2c}{\varphi_{min}(\Omega)}+2\mu+4\right)(c-1-c_2)$$

$$=\log(x_0^T Hx_0)+\frac{2c(c_2+1)}{\varphi_{min}(\Omega)}+\frac{8\gamma\log c}{2\gamma-\mu}-2(\tau+2)(c-1-c_2)$$

conversely ,

$$\frac{1}{t}\log\left(x^T(t)Hx(t)\right)\leq\frac{1}{c-1}(\log(x_0^T Hx_0))+\frac{2c(c_2+1)}{\varphi_{min}(\Omega)}+\frac{8\gamma\log c}{2\gamma-\mu}$$
$$-2(\tau+2)(c-1-c_2)$$

Then ,

$\lim_{t\to\to\infty}\frac{1}{t}\log\left(x^T(t)Hx(t)\right)\leq -2(\tau+2)$ (3.9) , $\forall\ \partial\in\varphi^*$

Hence , , $\forall\ \partial\in\varphi^*$, there exists a random number $C_3(\varphi)$ such that

$$\lim_{t\to\to\infty}\frac{1}{t}\log\left(x^T(t)Hx(t)\right)\leq -2(\tau+2)\quad ,\forall\ t\geq C_3$$

Thus ,

$$|x(t)|\leq\frac{e^{-(\tau+2)t}}{\sqrt{\varphi_{min}(\Omega)}}\qquad\forall\ \ t\geq C_3$$

By the convergence rate function , almost all $\partial\in\varphi^*$ such that

$$\int_0^\infty|h(t)x(t)|^n dt\leq\int_0^{c_3}N^n e^{n\tau t}|x(t)|^n dt+\int_0^\infty\frac{N^n e^{-nt}}{[\varphi_{min}(\Omega)]^{n/2}}dt<\infty \tag{4.0}$$

This contradicts the definition of φ^* . Hence , equation (4.0) satisfies the condition that
$\int_0^\infty|h(t)x(t,x_0)|^n dt<\infty$.

Theorem (5)

assume that $(H1)$ and $(H2)$ are satisfied for every $x_0 \in R^d$, either

$$\int_0^{\infty} |h(t)x(t)|^n dt \le \sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}} \qquad (5.1)$$

or

$$\lim_{t\to\infty} sup \frac{1}{t} \log(|x(t,x_0)|) < 0 \qquad (5.2)$$

Proof

Let $x(t) = x(t,x_0)$ $\forall\, x_0 \neq 0$. We only needs to show that the conclusion is true.

if $\varphi^* = \left\{\partial \in \varphi \colon \int_0^{\infty} |h(t)x(t)|^n dt > \sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}}\right\}$ $\qquad (5.3)$

We want to show that equation (5.2) holds for almost all $\partial \in \varphi^*$.

Let $\varphi_c^* = \left\{\partial \in \varphi \colon \int_0^{\infty} |h(t)x(t)|^n dt > (1+c)\sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}}\right\}$

for $c = 1, 2, 3$

suppose that $\varphi^* = \cup\varphi_c^*$, we show that for every $k \ge 1$, $\lim_{t\to\infty} sup \frac{1}{t} \log(|x(t,x_0)|)$

holds for almost all $\partial \in \varphi^*$. Let $k \ge 1$. For every $\partial \in \varphi - \varphi^*$, with φ^* and $p - null\ set$, there exists a random integer $C_4(\varphi)$ such that

$$\log\left(x^T(t)Hx(t)\right) \le \log(x_0^T H x_0) + \frac{2mt}{\varphi_{min}(\Omega)} + \frac{4\gamma(1+k^{-1})\log c}{2\gamma-\mu}$$

$$-\frac{2\gamma-\mu}{1+k^{-1}} \int_0^t \left(\int_0^s |h(u)x(u)|^n du\right)^2 ds \qquad (5.4)$$

$$\forall\ 0 \le t \le c, c \ge c_4 \quad .$$

Conversely , for every $\partial \in \varphi^*$, there exists a random number $C_5(\partial)$ such that

$$\int_0^t |h(s)x(s)|^n dt > (1+c^{-1})\sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}} \qquad (5.5)$$

, $\forall \ t \geq C_5$.

From equation (5.4) and (5.5) , we observed that almost all $\partial \in \varphi - \varphi^*$, if $c-1 \leq t \leq c$, $c \geq c_4 \vee (c_5+1)$.

$$\log\left(x^T(t)Hx(t)\right) \leq \log(x_0^T H x_0) + \frac{2c(c_5+1)}{\varphi_{min}(\Omega)} + \frac{4\gamma(1+c^{-1})\log c}{2\gamma-\mu} - \frac{2c}{c\varphi_{min}(\Omega)}(c-1-c_5)$$

Then the $\lim_{t\to\infty} sup \frac{1}{t}\log(|x(t,x_0)|) \leq -\frac{2c}{c\varphi_{min}(\Omega)} \ \ \forall \ \ \partial \in \varphi_c - \varphi^*$.

$$\lim_{t\to\infty} sup \frac{1}{t}\log(|x(t)|) \leq -\frac{c}{c\varphi_{min}(\Omega)} < 0 \ \ \forall \ \ \partial \in \varphi_c - \varphi^* .$$

Thus , if

$$\int_0^t |h(t)x(t,x_0)|^n dt > (1+c^{-1})\sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}} .$$

Hence , theorem (5) shows that if $\int_0^\infty |h(t)x(t,x_0)|^n dt > \sqrt{\frac{2N}{(2\gamma-\mu)\varphi_{min}(\Omega)}}$, then the solution of stochastic optimal control delay integro -differential equation tends to zero exponentially under additional condition and small time lag .

Conclusion .

In this study , We established the almost sure exponential stability of the non-linear stochastic optimal control integro- differential equations $(SOCIDDE_S)$ with constant delay or time lag . under lemma (3.1) , condition $(a) - (c)$ and H1 holds for the system (3.2) to be almost surely exponentially stable . Our findings reveal that , by replacing the noise scaling parameter of the stochastic system with finite integral expression and H2 , the system stabilized itself in an almost sure exponential sense . The sampled Lyapunov exponent must always kept finite for the stochastic system to be self stabilized in an exponential sense .

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