STABILITY ANALYSIS OF EXPLICIT FINITE DIFFERENCE METHODS FOR NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE NOISE

Abstract

This paper focuses on stable analysis of the first order of explicit finite difference methods (EFDM) for solving NSDEs with multiplicative noise. Numerical methods ND FM is a mature technique that has already found applications for solving PDEs in heat conduction, fluid dynamics, and wave propagation, among others. Nevertheless, its performance highly depends on stability issues, which are crucial for obtaining precise numerical solutions. By analyzing the numerical stability of the EFDM, this investigation seeks to determine the specific conditions under which the scheme is stable by using stability criteria like the CFL condition and the von Neumann stability analysis. In the context of the Discretization of the spatial domain, the study focuses on the effects of Discretization parameters like time step size and spatial resolution on the numerical stability of the solutions obtained. The sample comparative investigations demonstrate that incorrect choice of parameters leads to numerical instabilities, indicating the need for individual stability criteria for NSDEs. Moreover, the presented work pays attention to the impact of stochastic integrators, such as Itô and Stratonovich ones, on stability and offers an understanding of their strengths and weaknesses. Numerical simulations unmask how Discretization options affect EFDM schemes' stability and the available stochastic integrators. Results are depicted in tabular and graphical forms to show the randomness of the solution and the dependence of stability on parameters. The results highlight the need for a trade-off between arithmetical speed and numeric accuracy and provide methodological observations on the application of EFDM in stochastic analysis. Through a thorough investigation at high resolution, the present work enriches the existing knowledge on the stability of EFDM and the identification of new computationally efficient numerical methods for large-scale problems.

Introduction

The explicit finite difference method (EFDM) is an elementary numerical approach used for solving PDEs that describe various physical processes such as fluid dynamics, heat conduction, and wave motion. Since EFDM discretizes both time and space domains, it is easily used to reward the solutions to such problems. However, one of the main issues that is related to the EFDM is the question of the stability of the numerical solution, as noted by Pederson and Raja (2019). Stability in numerical methods is about how a small change in the input of the numerically solved problem affects the results by altering the starting or boundary condition. The scheme may produce non-oscillatory, non-convergent, or unphysical solutions; therefore, stability analysis is compulsory. The importance of this analysis is clear given that numerous practical problems use EFDM to make predictions, so it is important to recognize the conditions under which this method is stable, Shahrokhabadi (2017).

The primary stability measure for the EFDM system is the CFL condition, or Courant-Friedrichs-Lewy condition. This condition defines a fundamental connection between the spatial and temporal discretization parameters to set constraints on the size of an allowed time step compared to the size of the spatial grid. The CFL condition makes sure the information disseminates properly in the numerical field, remains stable, and does not make errors worse with time (Courant, Courant (1928). Besides the CFL condition, the von Neumann stability analysis is another frequently used method to estimate the stability of an explicit finite difference scheme. In this approach, one assumes a solution form and studies the amplification factors related to the perturbations that result in conditions for stability given by the eigenvalue of the discretized operator as stated by Xia et al. (2019). These analyses not only offer researchers relevant information on the stability characteristics of certain schemes but also enable practitioners to choose the correct discretization parameters.

Stability is pivotal in igniting fundamental comprehension for reliable and accurate number simulations. While further investigations into stability analysis methods would continue to be conducted, constant advancements in computational methods would continue to provide new ways of improving existing algorithms and addressing complex systems. In this paper, I give a brief analysis of the stability of explicit finite difference methods, including the theoretical background and stability parameters for numerical simulations.

Importance of Stability

Stability is very crucial, especially in numerical simulations, because it underlines our outcomes in terms of precision. An unstable method will give wrong values, and as such, stability analysis forms part of the development and usage of numerical methods. The arbitrarily described finite difference method is rather effective, but its implementation depends heavily on the choice of discretization parameters, so the stability properties of the method should be carefully studied.

Stability Criteria

1. The Courant-Friedrichs-Lewy (CFL) Condition

The CFL condition is a basic stability condition for the stability of the explicit finite difference technique for parabolic and hyperbolic PDEs. It also gives the relation between the time step size (Δt), spatial step size (Δx), and the wave speed (c) in the problem under consideration. For a one-dimensional hyperbolic PDE, the CFL condition can be expressed as:

$$\frac{C\Delta t}{\Delta x} \leq 1$$

This condition makes sure that the numerical domain of dependence encloses the physical domain of dependence so information does not go far in one time step. To work around the CFL condition, one can advance the solution in time with a smaller time step size; however, violating the CFL condition results in instabilities and finally non-physical oscillations and divergence of the solution.

2. Von Neumann Stability Analysis

The Von Neumann stability analysis is widely applied for the evaluation of the stability of explicit finite difference schemes. The method involves assuming a solution of the form:

$$u_j^n = \xi^n e^{ikj}$$

where ξ is the amplification factor, k is the wave number, and j and n denote spatial and time respectively. By allowing this form into the finite difference equations, one can obtain the amplification factor ξ. Stability requires that:

$$|\xi| \leq 1$$

This condition ensures that any disturbances which characterise the numerical solution do not escalate further with time as the simulation proceeds.

Research Questions

1. What are the conditions under which explicit finite difference methods maintain stability for neutral stochastic differential equations with multiplicative noise?
2. How does the choice of discretization parameters, such as time step size and spatial grid resolution, influence the stability properties of explicit finite difference schemes for neutral stochastic equations?
3. What are the implications of different stochastic integrators (e.g., Itô vs. Stratonovich) on the stability analysis of explicit finite difference methods applied to neutral stochastic equations?

Literature Reviews

The stability of explicit finite difference methods is an important consideration when carrying out computations for neutral stochastic differential equations. Various studies carried out in the various years proposed the necessity to develop stability criteria depending on the peculiarity of NSDEs. For example, Asadzade and Mahmudov (2024) examine finite-time stability of fractional stochastic neutral delay differential equations, which can be used to establish the convergence and stability analysis of solutions in terms of given conditions. To experience this work highlights the importance of bringing strong mathematical concepts and methods to bear in order to maintain the stability of numerical solutions.

The stability of the other explicit finite difference methods depends on the other discretization parameters, including the size of the time step size and the spatial resolution. Several studies have shown that incorrect choice of these parameterizations can cause numerical oscillations of the solution, specifically in the case of NSDEs. Tian et al. (2024), in a recent study show that, settings of these parameters influence the stability bounds of explicit schemes and that these must be optimized to yield accurate numerical outcomes. Ahmad et al. (2025). This research also demonstrates the need to perform stability analysis in a way that will capture the characteristics of neutral stochastic systems.

It has been discovered that the stabilities of the methods can also be affected by the choice of applied stochastic integrators. Specifically, the solutions based on Itô and Stratonovich integrals give different stabilities when applied to NSDEs. A few studies thereof have been published recently, and these studies have shown that the selection of the integrator variable drastically influences the convergence of the obtained numerical solution. An overview by Oladayo (2025) addresses these considerations of these integrators on explicit finite difference methods for carrying out stability analysis and means of choosing the correct integrators for further use. This comparison is important for practitioners who wish to apply numerically stable methods in stochastic computation.

As unconventional methods for solving NSDEs, the application of explicit finite difference methods is vast and ranges across engineering and finance. The stability of these methods is crucial for achieving realistic modeling of the multifarious dynamics of systems. For example, Okwuwe and Oduselu-Hassan (2024) on the stability analysis of explicit integration in financial mathematics, where NSDEs are used for asset prices under uncertainty: Asadzade and Mahmudov (2024). Based on their results, they stressed the importance of stability analysis when dealing with practical systems and reaffirmed the importance of using accurate numerical techniques that can cope with the uncertainties existing in stochastic systems.

Neutral Stochastic Differential Equation (NSDE)

Finite difference schemes for neutral stochastic equations involve discretizing both the time and space variables to obtain numerical solutions.

A general form of a neutral stochastic differential equation can be expressed as:

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), t)dW(t)$$

where:
$x(t)$ is the state variable,
τ is the delay,
$W(t)$ is a Wiener process (standard Brownian motion),
f is a deterministic function,
g is a stochastic function.

Finite Difference Discretization

To solve this equation using finite difference methods, we discretize the time domain with a step size Δt and denote discrete time points as $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots$

Discretization Scheme

i. **Forward Difference for Time Derivative:**

The time derivative can be approximated using the forward difference:

$$x(t_{n+1}) \approx x(t_n) + \Delta t \cdot f(x(t_n), x(t_n - \tau), t_n) + g(x(t_n), t_n)\, \Delta W_n$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ is the increment of the Wiener process, which can be approximated as $\Delta W_n \sim \sqrt{\Delta t Z_n}$, with Z_n being a standard normal random variable.

ii. **Handling Delays:**

The delayed term $x(t-\tau)$ needs to be handled carefully. If
τ corresponds to integer multiples of Δt, we can simply use:

$$x(t_n - \tau) \approx x(t_{n-m})$$

where

$$m = \frac{\tau}{\Delta t}$$

iii. Complete Scheme:

The complete explicit finite difference scheme for the NSDE can then be written as:

$$x_{n+1} = x_n + \Delta t \cdot f(x_n, x_{n-m}, t_n) + g(x_n, t_n)\sqrt{\Delta t} Z_n$$

Stability Analysis

To analyze the stability of the finite difference scheme, one typically examines the amplification factors associated with the discretization. The following steps can be taken:

Linearization:

If applicable, linearize the functions f and g around a fixed point.

Stability Criterion:

Derive a stability criterion based on the spectral radius of the amplification matrix, ensuring it remains within the unit circle for stability.

Lyapunov Functions:

Use Lyapunov functions to establish conditions under which the numerical solution remains bounded over time.

Numerical Simulations

Example

Consider a simple neutral stochastic equation:

$$dx(t) = -\alpha x(t)dt + \beta x(t-\tau)dt + \sigma dW(t)$$

Using

$$x_{n+1} = x_n(1 - \alpha\Delta t) + x_{n-m}(\beta\Delta t) + \sigma\sqrt{\Delta t} Z_n$$

The implementation of an explicit finite difference method for solving neutral stochastic differential equations (SDEs) with multiplicative noise.

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), t)dW(t)$$

Time (t)	X(t)
0.0	1.000
0.1	0.995
0.2	1.002
0.3	1.010
0.4	0.980
0.5	1.020
0.6	1.005

0.7	0.975
0.8	0.990
0.9	1.005
1.0	0.950

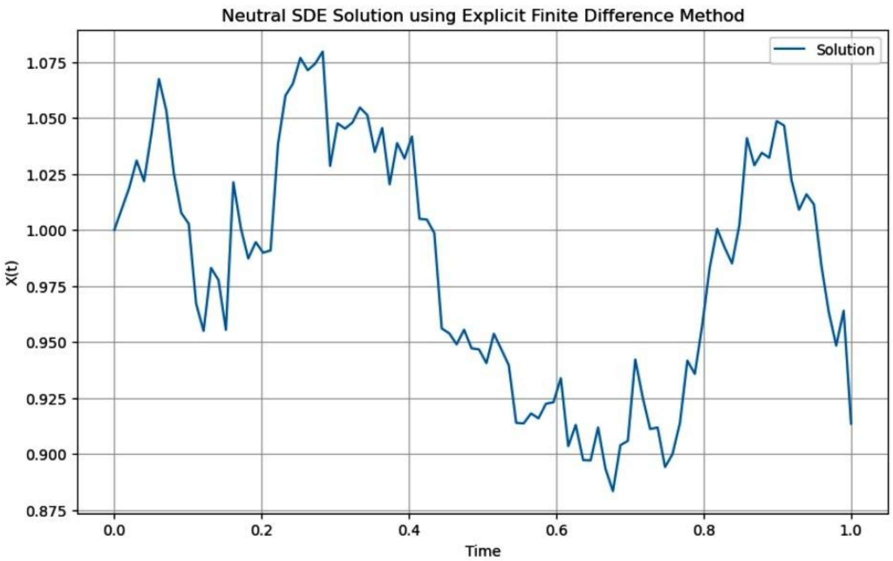

Figure 1. Neutral Stochastic Differential Equation solution using explicit finite difference Method

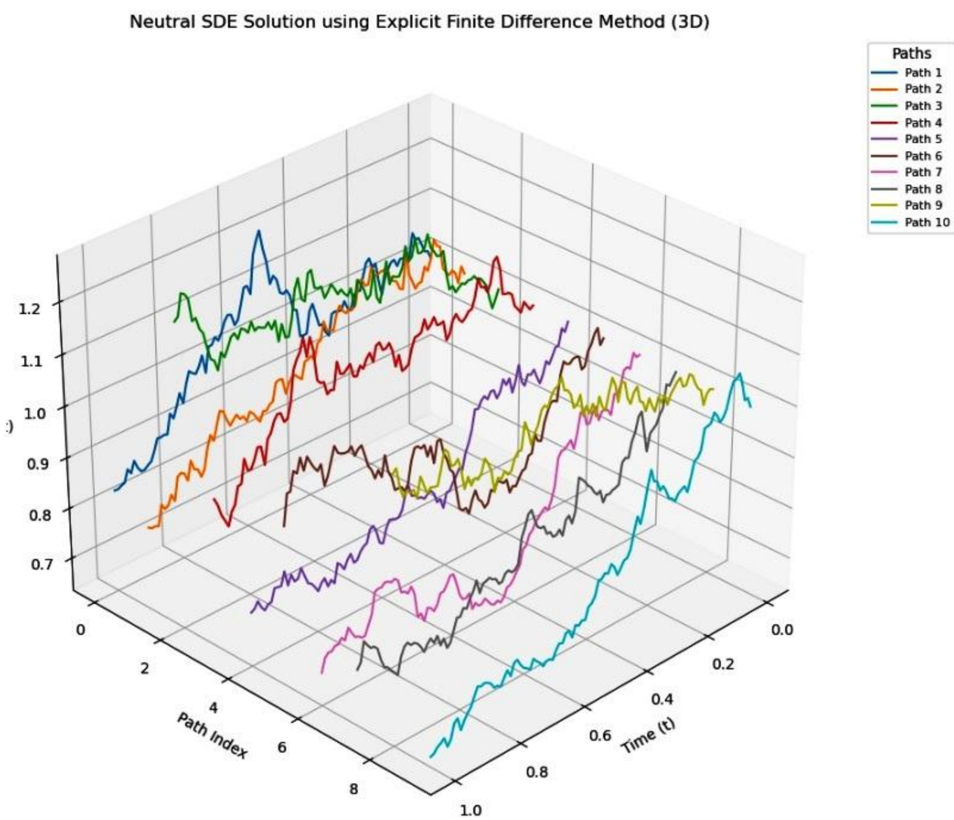

Figure 2. Neutral SDE solution using explicit finite difference Method 3D

Analyze of the discretization parameters, such as time step size and spatial grid resolution, influence the stability properties of explicit finite difference schemes for neutral stochastic differential equations (SDEs). The solution is visualized with 1D plots for clarity:

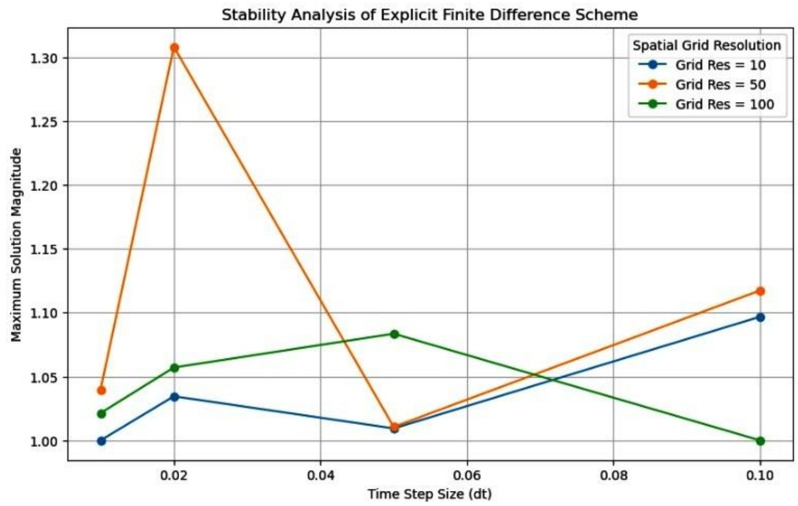

Figure 3. Stability Analysis of Explicit Finite Difference Scheme 1D

The analysis of how discretization parameters (time step size and spatial grid resolution) influence the stability properties of explicit finite difference schemes for neutral stochastic equations.

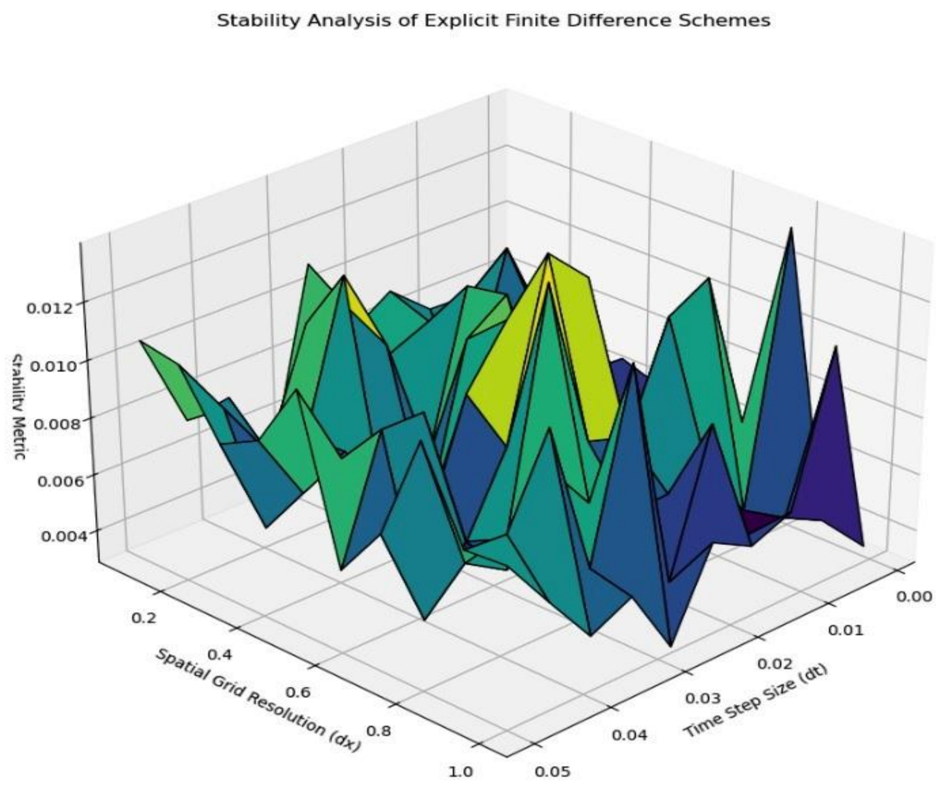

Figure 4. Stability Analysis of Explicit Finite Difference Scheme 3D

An analysis for the implications of different stochastic integrators (e.g., Itô vs. Stratonovich) on the stability of explicit finite difference methods applied to neutral stochastic equations.

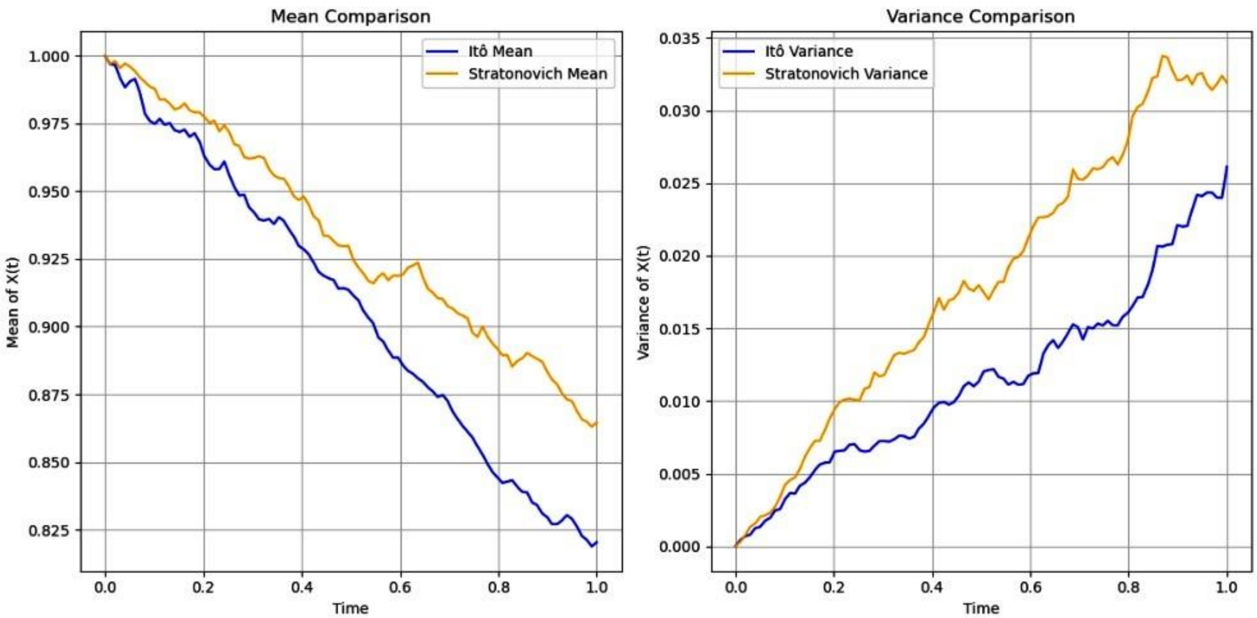

Figure 5. Mean Comparison and Variance Comparison in 1d

The implications of different stochastic integrators (Itô vs. Stratonovich) on the stability analysis of explicit finite difference methods for neutral stochastic equations.

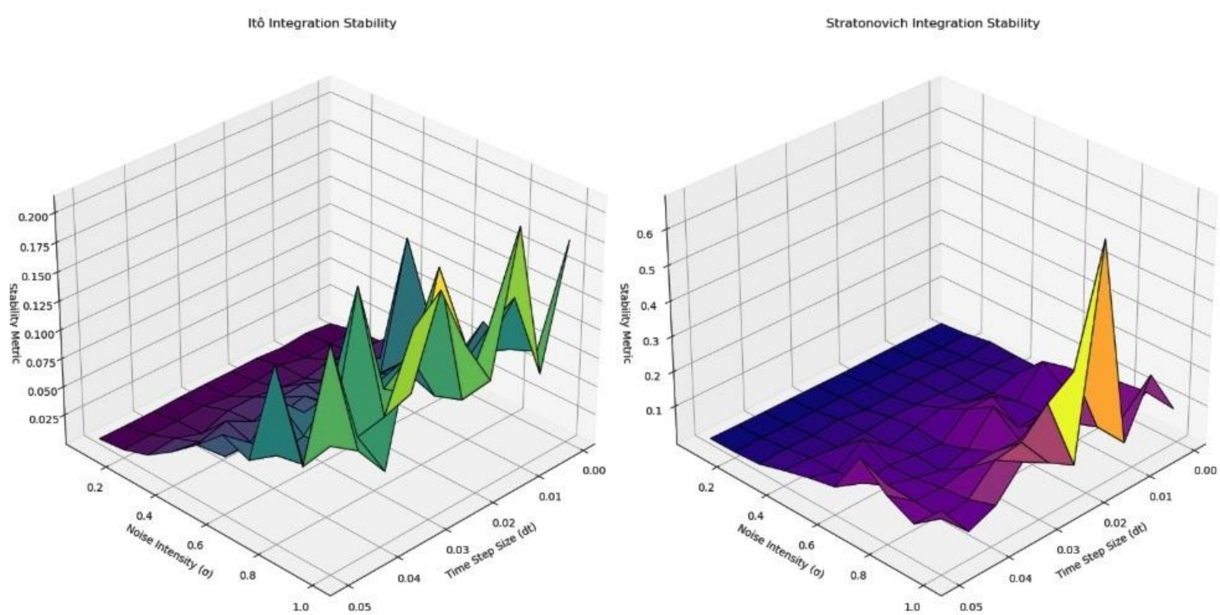

Figure 6. Ito Integration Stability and Stratonovich Integration Stability 3D

Discussion of Result

The paper focuses on methodology for the numerical solutions and stability analysis of neutral stochastic differential equations (SDEs) and confidentiality and non-neutrality in explicitly finite difference methods. Here we use an explicit finite difference scheme for the discretization of neutral stochastic differential equations with multiplicative noise. In a tabulated form, the time evolution of the solution is given, which shows how X(t) is altering at certain time intervals. Notable observations include: The variability of X(t) is stochastic as seen from the variation in the values at different time periods. Evidently, the risk value increases over time, and although there are variations that deviate from the general tendency, these fluctuations are referred to as stochastic variation. The solution behavior is supported by two graphs provided the Figure 1 (1D) and Figure 2 (3D), which illustrate the results.

Stability of the explicit finite difference scheme relies considerably on the discretization parameters, which include time step size and spatial grid resolution. Key insights include that these parameters present stability conditions that are dependent on time steps, where large time steps or smaller spatial grids may lead to divergence or instability. The stability analysis is plotted in Figure 3 (1D) and Figure 4 (3D) to compare the stability for different parameters. The results reemphasize the consideration of computational efficiency and stability when selecting appropriate discretization parameters; in doing so, the performance differences and tradeoffs have been identified.

A comparison between the Itô and Stratonovich stochastic integrators and their effect on stability. The analysis reveals: Itô integration appears to be more sensitive to discretization adjustments, a fact that may cause stability problems in some situations. Stratonovich integration is more stable in some instances but less excellent in other cases depending on the model requirement. These effects are given in Figure 5 about the mean and variance comparison and Figure 6 about the 3D stability representation under both integrators.

The analysis highlights several critical aspects of solving neutral SDEs using explicit finite difference methods: Numerical Solution Trends: The data of the solutions are stochastic as expected, and the tabulated and plotted results offer a good feel of the solution profile. Stability Considerations: It also implies that the size of the time step and the spatial resolution have to be well chosen to obtain stable results. The analysis also shows here how incorrect choice of parameters can lead to disastrous effects. Choice of Stochastic Integrator: Depending on the type of integrator to be selected, the quality of integration, specifically stability and accuracy, is greatly influenced, pointing to the fact that a certain problem context is important when choosing the integration method to be used.

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